Generalized Calogero-Moser spaces and rational Cherednik algebras

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For my family.

We shall not cease from exploration.
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.

T.S. Eliot, Little Gidding.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. The work has not been submitted for any other degree or professional qualification.

(*Gwyn Bellamy*)

Summary

The subject of this thesis is the interplay between the geometry and the representation theory of rational Cherednik algebras at $t = 0$. Exploiting this relationship, we use representation theoretic techniques to classify all complex reflection groups for which the geometric space associated to a rational Cherednik algebra, the generalized Calogero-Moser space, is singular. Applying results of Ginzburg-Kaledin and Namikawa, this classification allows us to deduce a (nearly complete) classification of those symplectic reflection groups for which there exist crepant resolutions of the corresponding symplectic quotient singularity.

Then we explore a particular way of relating the representation theory and geometry of a rational Cherednik algebra associated to a group W to the representation theory and geometry of a rational Cherednik algebra associated to a parabolic subgroup of W. The key result that makes this construction possible is a recent result of Bezrukavnikov and Etingof on completions of rational Cherednik algebras. This leads to the definition of cuspidal representations and we show that it is possible to reduce the problem of studying all the simple modules of the rational Cherednik algebra to the study of these finitely many cuspidal modules. We also look at how the Etingof-Ginzburg sheaf on the generalized Calogero-Moser space can be "factored" in terms of parabolic subgroups when it is restricted to particular subvarieties. In particular, we are able to confirm a conjecture of Etingof and Ginzburg on "factorizations" of the Etingof-Ginzburg sheaf.

Finally, we use Clifford theoretic techniques to show that it is possible to deduce the Calogero-Moser partition of the irreducible representations of the complex reflection groups $G(m, d, n)$ from the corresponding partition for $G(m, 1, n)$. This confirms, in the case $W = G(m, d, n)$, a conjecture of Gordon and Martino relating the Calogero-Moser partition to Rouquier families for the corresponding cyclotomic Hecke algebra.

Introduction

In this thesis we investigate a class of algebras called rational Cherednik algebras. These algebras, which were introduced by Etingof and Ginzburg [36] as part of a more general class of algebras called symplectic reflection algebras, are related to an astonishingly large number of apparently disparate areas of mathematics such as combinatorics, integrable systems, real algebraic geometry, quiver varieties, resolutions of symplectic singularities and, of course, representation theory. As such, their exploration entails a journey through a colourful and exciting landscape of mathematical constructions. In particular, as we hope to illustrate throughout this body of work, studying rational Cherednik algebras involves a deep interplay between geometry and representation theory.

Let W be an irreducible complex reflection group acting on a reflection representation $\mathfrak{h}$. The *rational Cherednik algebras* $H_{t,\mathbf{c}}(W)$, associated to W and $\mathfrak{h}$, are a flat family of deformations of the skew group ring $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes W$, depending on parameters t and $\mathbf{c}$. When $t = 0$, the algebras are finite modules over their centres $Z_{\mathbf{c}}$ and thus their representation theory is intimately linked to the geometry of the centre. The affine variety corresponding to $Z_{\mathbf{c}}$ is called the *generalized Calogero-Moser space* $X_{\mathbf{c}}$. One of the most basic questions one can ask about the space $X_{\mathbf{c}}(W)$ is: *for which values of the parameter* $\mathbf{c}$ *is* $X_{\mathbf{c}}(W)$ *smooth?* The complete answer to this question is not known. However, in the first part of this thesis we give a complete classification of those complex reflection groups whose corresponding generalized Caloger-Moser space is a singular variety for all values of the parameter $\mathbf{c}$. Then, using work of Ginzburg-Kaledin [45] and Namikawa [78], we are able to relate this to the question of the existence of symplectic resolutions for the symplectic quotient varieties $\mathfrak{h} \times \mathfrak{h}^*/W$. To say that a variety is symplectic roughly means that the smooth locus of $\mathfrak{h} \times \mathfrak{h}^*/W$ is a symplectic manifold with closed 2-form ω and we say that there exists a symplectic resolution π of $\mathfrak{h} \times \mathfrak{h}^*/W$ if

$$\pi : X \longrightarrow \mathfrak{h} \times \mathfrak{h}^*/W$$

is a resolution of singularities such that $\pi^*\omega$ extends to a non-degenerate symplectic 2-form on the whole of X. For an arbitrary symplectic variety it is not know if such resolutions exist. However, using our classification result we are able to fully answer this question for the cases described above. In particular, we show that there exists a symplectic resolution of the four dimensional quotient singularity $\mathfrak{h} \times \mathfrak{h}^*/G_4$, where G_4 is the binary tetrahedral group (see section 4.3 for details). This is a new example of a symplectic resolution.

We then look at a way of relating the representation theory and geometry of a rational Cherednik algebra associated to a group W to the representation theory and geometry of a rational Cherednik algebra associated to a parabolic subgroup of W. The key result that makes this analysis possible is a recent construction of Bezrukavnikov and Etingof [10]. They show that a certain completion of the rational Cherednik algebra is isomorphic to the ring of matrices over a completion of a rational Cherednik algebra associated to a parabolic subgroup. Using this result we show that there is a "factorization" of certain closed subvarieties of $X_{\mathbf{c}}(W)$. To be specific, there exists a surjective morphism

$$\pi_W \,:\, X_{\mathbf{c}}(W) \twoheadrightarrow \mathfrak{h}/W,$$

and to each point $b \in \mathfrak{h}/W$ we associate a parabolic subgroup W_b of W such that

$$\Phi \,:\, \pi_W^{-1}(b) \xrightarrow{\sim} \pi_{W_b}^{-1}(0).$$

Since W_b is often a product irreducible complex reflection groups, we say that the closed subvariety $\pi_W^{-1}(b)$ of $X_{\mathbf{c}}(W)$ has been "factorized" into a product of smaller dimensional varieties. We then consider a certain sheaf on the generalized Calogero-Moser space, the *Etingof-Ginzburg sheaf.* Denoting by $\mathbf{e}$ the trivial idempotent of W, the Etingof-Ginzburg sheaf $\mathcal{R}[W]$ is the sheaf corresponding to the $Z_{\mathbf{c}}$-module $H_{0,\mathbf{c}} \cdot \mathbf{e}$. In the case $W = S_n$, the symmetric group, it was conjectured by Etingof and Ginzburg that their sheaf $\mathcal{R}$ should also "factorize" as a S_n-equivariant sheaf in some precise sense. We show that such a factorization actually exists for all irreducible complex reflection groups, that is there exists an isomorphism of W-equivariant sheaves:

$$\Phi_* \left(\mathcal{R}[W]|_{|_{\pi_W^{-1}(b)}} \right) \simeq \mathtt{Ind}\,_{W_b}^{W} \mathcal{R}[W_b]|_{|_{\pi_{W_b}^{-1}(0)}}.$$

A second application of Bezrukavnikov and Etingof's result is to do with those finite dimensional quotients of $H_{\mathbf{c}}(W)$ that are supported on a closed point of $X_{\mathbf{c}}(W)$. Let $\chi \in X_{\mathbf{c}}(W)$ and $H_{\mathbf{c},\chi} := H_{0,\mathbf{c}}/\mathfrak{m}_\chi \cdot H_{0,\mathbf{c}}$ be the "largest" quotient of $H_{0,\mathbf{c}}$ supported at χ (here $\mathfrak{m}_\chi$ is the maximal ideal of $Z_{\mathbf{c}}$ defining χ). The space $X_{\mathbf{c}}(W)$ has a stratification by symplectic leaves and it has been shown by Brown and Gordon that $H_{\mathbf{c},\chi_1} \simeq H_{\mathbf{c},\chi_2}$ if χ_1 and χ_2 lie on the same leaf of $X_{\mathbf{c}}(W)$. Let $\mathcal{L}$ denote the symplectic leaf on which χ sits. If $\mathcal{L}$ is a zero-dimensional leaf, $\mathcal{L} = \{\chi\}$, then we call $H_{\mathbf{c},\chi}$ a *cuspidal algebra.* Our main result in this direction is:

Theorem. Let $\mathcal{L}$ be a leaf in $X_{\mathbf{c}}(W)$ of dimension $2l$ and χ a point on $\mathcal{L}$. Then there exists a parabolic subgroup W_b, $b \in \mathfrak{h}$, of W of rank $\dim \mathfrak{h} - l$ and a cuspidal algebra $H_{\mathbf{c}',\psi}$ with $\psi \in X_{\mathbf{c}'}(W_b)$ such that

$$H_{\mathbf{c},\chi} \simeq \mathrm{Mat}_{\,|W/W_b|}\,(H_{\mathbf{c}',\psi}).$$

As a consequence we show that there exists a functor

$$\Phi_{\psi,\chi} \,:\, H_{\mathbf{c}',\psi}\text{-mod} \xrightarrow{\sim} H_{\mathbf{c},\chi}\text{-mod}$$

defining an equivalence of categories such that

$$\Phi_{\psi,\chi}(M) \simeq \mathsf{Ind}\,_{W_b}^{W} M \quad \forall\, M \in H_{\mathbf{c}',\psi}\text{-mod}$$

as W-modules. This shows that the problem of describing the W-module structure of the simple $H_{0,\mathbf{c}}(W)$-modules reduces to studying the simple modules of the cuspidal algebras.

There is a canonically defined finite dimensional quotient of the rational Cherednik algebra called the *restricted rational Cherednik algebra* . This algebra has a very combinatorially rich representation theory. One of the most elementary questions one can ask is to describe the blocks of this algebra. These blocks define a partition of the set of simple modules, called the *Calogero-Moser partition.* Using the geometry of a resolution of the singular space $\mathbb{C}^{2n}/C_m \wr S_n$, Gordon and Martino gave a combinatorial description of the Calogero-Moser partition when W is the wreath product $C_m \wr S_n$. Using Clifford theoretic arguments we extend this to a description of the Calogero-Moser partition when W is the normal subgroup $G(m,d,n)$ of $C_m \wr S_n$. By comparing our answer with the description of Rouquier families given by Chlouveraki, this allows us to confirm in the case $W = G(m,d,n)$ a conjecture of Gordon and Martino relating the Calogero-Moser partition to Rouquier families for cyclotomic Hecke algebras.

Structure of the thesis

The structure of this thesis is as follows. In chapter one we define Poisson manifolds and Poisson varieties and list some of their fundamental properties. We also introduce the skew group-ring. The star of the show, the rational Cherednik algebra, is introduced in chapter two. Fundamental properties of the algebra, due to Etingof and Ginzburg, are presented and the restricted rational Cherednik algebra is also introduced. Chapter three contains the classification of those complex reflection groups for which the associated generalized Calogero-Moser space is always singular. We show in chapter four, using results of Ginzburg-Kaledin and Namikawa, that this classification can be used to give a (nearly complete) classification of those symplectic reflection groups for which there exists crepant resolutions of the corresponding symplectic quotient singularity. Then chapter five looks at a way of relating the representation theory and geometry of a rational Cherednik algebra associated to a group W to the representation theory and geometry of a rational Cherednik algebra associated to a parabolic subgroup of W. In chapter six we study the Calogero-Moser partition for the groups $G(m,d,n)$, extending the results of Gordon and Martino for $G(m,1,n)$. Finally, chapter seven compares our description of the Calogero-Moser partition for $G(m,d,n)$ given in chapter six with the description of Rouquier families for cyclotomic Hecke algebras as given by Chlouveraki.

Contents

Chapter 1

Poisson manifolds and orbifolds

1.1 Notation

We begin by fixing the notation that will be used throughout this thesis. Unless otherwise stated, a module will always refer to a left module. An algebra refers to a unital ring A that is a $\mathbb{C}$-vector space such that the image of the natural map $\mathbb{C} \to A$, $a \mapsto a \cdot 1_A$ is central. If $B \subset A$ is a subalgebra then we do not assume that the unit in B equals the unit in A. An *affine* algebra will always refer to a commutative, finitely generated $\mathbb{C}$-algebra. If A is a (not necessarily commutative) algebra and Z a central subalgebra of A then we say that A is finite over Z if A is a finite Z-module. A manifold will always mean a finite dimensional complex analytic manifold with a holomorphic sheaf of functions.

1.2 Poisson manifolds

Poisson structures arise naturally in the study of noncommutative algebras. Therefore we begin by introducing Poisson manifolds and stating some of their basic properties.

Definition 1.1. Let M be a manifold and $\mathcal{O}_M$ the sheaf of holomorphic function on M. Then M is said to be a *Poisson manifold* if there exists a bilinear map (the *Poisson bracket*) $\{-,-\} : \mathcal{O}_M \times \mathcal{O}_M \to \mathcal{O}_M$ such that

1. for each open subset $U \subset M$, $(\mathcal{O}_M(U), \{-,-\})$ is a Lie algebra,
2. for $V \subset U \subset M$ open sets, the structure map ρ_{UV} is a Lie algebra morphism,
3. for $f \in \mathcal{O}_M(U)$, the map $\{f,-\} : \mathcal{O}_M(U) \to \mathcal{O}_M(U)$ is a derivation.

Note that (3) of the above definition simply says that every function $f \in \mathcal{O}_M(U)$ defines a vector field ξ_f on the open set U with $\xi_f \cdot g := \{g, f\}$ for all $g \in \mathcal{O}_M(U)$. A vector field ν on U is called a Hamiltonian vector field if there exists some $f \in \mathcal{O}_M(U)$ such that $\nu = \xi_f$. Therefore the Poisson bracket defines a map $\mathcal{O}_M \to \Theta_M$ from the structure sheaf on M into the sheaf of vector fields on M.

It is possible to encode the data of a Poisson bracket on M as a 2-vector field. If Θ_M denotes the sheaf of vector fields on M, then the sheaf of i-vector fields, Θ_M^i, on M is the sheaf $\bigwedge^i(\Theta_M)$. There is a natural pairing between i-vector fields and i-forms, $\langle -,- \rangle : \Theta_M^i \times \Omega_M^i \to \mathcal{O}_M$ defined by

$$\langle X_1 \wedge \cdots \wedge X_i, d_1 \wedge \cdots \wedge d_i \rangle = \sum_{\pi \in S_i} sgn(\pi) d_1(X_{\pi(1)}) \cdot \cdots \cdot d_i(X_{\pi(i)}).$$

Given a 2-vector field $\Pi \in \Theta_M^2(M)$, one can define a "bracket" on M by $\{f,g\}_\Pi := \langle \Pi, df \wedge dg \rangle$. However, for an arbitrary element Π, this bracket will not satisfy the Jacobi identity. The requirement that $\{-,-\}_\Pi$ satisfies the Jacobi identity can be concisely written in terms of the *Schouten bracket* on $\Theta_M^\bullet$ (see [33, §1.8]): $\{-,-\}_\Pi$ satisfies the Jacobi identity if and only if $[\Pi,\Pi] = 0$. Since the pairing $\langle -,- \rangle$ is perfect, every Poisson bracket can be written as $\{-,-\}_\Pi$ for some $\Pi \in \Theta_M^2(M)$.

Example 1.2. Let (M,ω) be a symplectic manifold, where ω is a non-degenerate closed 2-form. If X is a vector field on M then define $\omega^\#(X) := i_X\omega$, the contraction of ω by X (where $i_X\omega(Y) = \omega(X,Y)$). The non-degeneracy of ω means that the corresponding homomorphism $\omega^\# : TM \to T^*M$ is an isomorphism. To each $f \in \mathcal{O}_M$ we can associate a vector field X_f as follows: X_f is defined to be the unique vector field on M such that $i_{X_f}\omega = -df$. This allows us to define a bracket on M, the Poisson bracket of ω, as follows:

$$\{f,g\} := \omega(X_f, X_g) = X_f(g) = -X_g(f).$$

Then one can check (see [33, Proposition 1.1.7]) that $\{-,-\}$ is actual a Poisson bracket on M. The fact that $\{-,-\}$ satisfies the Jacobi identity is equivalent to the fact that ω is closed. From the definition of $\{-,-\}$ we see that the vector field X_f is the Hamiltonian vector field of $f \in \mathcal{O}_M$.

Let X be a vector field on M and $p \in M$. An *integral flow* for X through p is a holomorphic function $\rho : B \to M$, where $B = \{t \in \mathbb{C} \mid |t| < \varepsilon\}$ for some $\varepsilon > 0$, such that

$$X(f)(\rho(t)) = \frac{d\rho}{dt}(f)(t) \quad \forall f \in \mathcal{O}_M(U),\, U \subset M \text{ open},\, t \in \rho^{-1}(U).$$

As shown in [1, Chapter 3], there always exists an integral flow for X through each point p and any two integral flows will agree on their common domain of definition. If $f \in \mathcal{O}_M(U)$ and X_f is the corresponding Hamiltonian vector field then an integral flow for X_f is called a *Hamiltonian flow.*

Using Hamiltonian flows we can introduce symplectic leaves. These, at least in their algebraic formulation which will be given below, play an important role in the study of the representation theory of rational Cherednik algebras. We define an equivalence relation on M by saying that $p \sim q$ if q can be reached from p by a piecewise smooth curve, each segment of which is a Hamiltonian flow. The equivalence classes of this relation are called the symplectic leaves of M. Their properties are summarized in [101, Proposition 1.3], where the result is attributed to Kirillov [68] but see also [33, §1.5].

Proposition 1.3 (Proposition 1.3, [101]). *The symplectic leaves of M are connected Poisson submanifolds and the dimension of each such submanifold $\mathcal{L}$ equals the rank of the Poisson bracket at each point* *of $\mathcal{L}$.*

If M is a symplectic manifold then the Poisson bracket is everywhere non-degenerate. Therefore the whole manifold M is a symplectic leaf.

Example 1.4. Let G be a finite dimensional, connected, complex Lie group and $(\mathfrak{g}, [-,-])$ its Lie algebra. Then the symmetric algebra on $\mathfrak{g}$ inherits a Poisson structure from the Lie bracket on $\mathfrak{g}$ if we define $\{X, Y\} := [X, Y]$ for $X, Y \in \mathfrak{g} \subset S(\mathfrak{g})$ (since $\mathfrak{g}$ generates $S(\mathfrak{g})$ as an algebra this uniquely defines a Poisson bracket on $S(\mathfrak{g})$) and hence $\mathfrak{g}^*$ is a Poisson manifold. The bracket on $\mathfrak{g}^*$ is commonly known as the *Kostant-Kirillov-Souriau* bracket. The group G acts via the adjoint action on $\mathfrak{g}$: for $g \in G$, $Ad_g : \mathfrak{g} \to \mathfrak{g}$, $X \mapsto g^{-1} \cdot X \cdot g$. Then G also acts on $\mathfrak{g}^*$ via the coadjoint action, $Ad_g^*(\alpha)(X) = \alpha(Ad_g^{-1}(X))$ where $g \in G, X \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$. One can give a concise description of the symplectic leaves of $\mathfrak{g}^*$. We follow the proof given in [33, Theorem 1.5.8] but see also [25, Proposition 1.3.21].

Lemma 1.5. *The symplectic leaves of $\mathfrak{g}^*$ are precisely the coadjoint orbits of G.*

Proof. The Leibniz rule for differentiation implies that the tangent space to a point of a leaf of $\mathfrak{g}^*$ is spanned by the Hamiltonian vector fields corresponding to the linear functions on $\mathfrak{g}^*$. The linear functions on $\mathfrak{g}^*$ are, by definition, the elements of $\mathfrak{g}$. This means that the tangent spaces of the symplectic leaves equals the tangent spaces of the coadjoint orbits. Therefore the coadjoint orbits are open, closed subsets of the symplectic leaves. Therefore they must coincide with the symplectic leaves of $\mathfrak{g}^*$ because the symplectic leaves are connected by definition and we have assumed that our group G is connected. $\square$

1.3 Poisson varieties

Let Z be an affine algebra. It is said to be a *Poisson algebra* if there exists a bilinear map, the Poisson bracket, $\{-,-\} : Z \times Z \to Z$ such that $(Z, \{-,-\})$ is a Lie algebra and $\{f, -\}$ defines a derivation of Z for each $f \in Z$. The affine variety corresponding to a Poisson algebra is called a Poisson variety. If S is a multiplicatively closed set in Z and Z_S the localization of Z at S then one can use the "quotient rule" in differentiation to extend the Poisson bracket to Z_S. This means that a Poisson bracket on Z defines a Poisson bracket on the sheaf of regular (algebraic!) functions on $X := \mathrm{Spec}\,(Z)$. An ideal I in Z is called a *Poisson ideal* if $\{I, Z\} \subset I$. If I is a Poisson ideal then the quotient algebra Z/I inherits a Poisson bracket from Z: $\{z_1 + I, z_2 + I\} := \{z_1, z_2\} + I$. We call a prime ideal that is Poisson a *Poisson prime.*

Let X now stand for $\mathrm{Maxspec}\,(Z)$. Following [17, Section 3.2], we define the *Poisson core* of an ideal J of Z to be the largest Poisson ideal of Z contained in J, and denoted it by $\mathcal{C}(J)$. It exists because the sum of two Poisson ideals is again a Poisson ideal. If J is prime then $\mathcal{C}(J)$ is also prime and when $\mathfrak{m}$ is maximal, $\mathcal{C}(\mathfrak{m})$ is said to be *Poisson primitive.* We say that $\mathfrak{m}$ is *maximal Poisson* if it is a maximal ideal of Z that is Poisson. Clearly, every maximal Poisson ideal is Poisson primitive. Using Poisson cores we can define an equivalence relation on X: $\mathfrak{m} \sim \mathfrak{n}$ if and only if $\mathcal{C}(\mathfrak{m}) = \mathcal{C}(\mathfrak{n})$. We denote by $\mathcal{Q}(\mathfrak{m})$ the equivalence class of $\mathfrak{m}$, so that

$$X = \bigsqcup \mathcal{Q}(\mathfrak{m})$$

is a stratification of X by symplectic cores. As noted in [17, Lemma 3.3] the sets $\mathcal{Q}(\mathfrak{m})$ are locally closed. However it is not known whether $\overline{\mathcal{Q}(\mathfrak{m})} = V(\mathcal{C}(\mathfrak{m}))$, see [17, Question 3.2]. Note that, in general, this stratification is not finite. For example, let $Z = \mathbb{C}[x, y, c]$ with Poisson bracket $\{c, x\} = \{c, y\} = 0$ and $\{x, y\} = 1$. Then the Poisson cores of Z are $(c - \alpha) : \alpha \in \mathbb{C}$ and (0). Below we will define symplectic leaves for Poisson varieties. In many cases, including rational Cherednik algebras, the stratification by symplectic leaves agrees with the stratification by Poisson cores and this purely algebraic interpretation of leaves will be very useful.

One way of creating interesting new examples of Poisson algebras is through quantization. Let A be a $\mathbb{C}$-algebra, $\mathbf{t}$ a central non-zero divisor and $\rho : \mathsf{A} \twoheadrightarrow A := \mathsf{A}/\mathbf{t} \cdot \mathsf{A}$ the quotient map. Assume that there exists an affine central subalgebra Z of A and $\mathsf{Z} \subset \mathsf{A}$ such that ρ induces an isomorphism $\mathsf{Z}/\mathbf{t} \cdot \mathsf{Z} \simeq Z$. Let $\{z_i : i \in I\}$ be a $\mathbb{C}$-basis for Z and choose a lift $\hat{z}_i$ of z_i in Z for every $i \in I$. As explained in [17, (2.2)], one has

Lemma 1.6. *The rule*

$$\{z_i, z_j\} = \rho([\hat{z}_i, \hat{z}_j]/\mathbf{t})$$

extends by linearity to a Poisson bracket on Z.

Proof. Let us first check that the binary operation is well-defined. If $\hat{z}_1, \hat{z}_2$ are any two elements in Z then $\rho([\hat{z}_1, \hat{z}_2]) = [\rho(\hat{z}_1), \rho(\hat{z}_2)] = 0$. Therefore there exists $\hat{z}_3 \in \mathsf{Z}$ such that $[\hat{z}_1, \hat{z}_2] = \mathbf{t} \cdot \hat{z}_3$. Since $\mathbf{t}$ is a non-zero divisor, $\hat{z}_3$ is unique. Hence the expression $\rho([\hat{z}_1, \hat{z}_2]/\mathbf{t})$ is well-defined. The fact that $\rho([\mathbf{t} \cdot \hat{z}_1, \hat{z}_2]/\mathbf{t}) = [\rho(\hat{z}_1), \rho(\hat{z}_2)] = 0$ shows that the bracket is independent of choice of lift. The fact that the bracket makes Z into a Lie algebra and satisfies the derivation property is a consequence of the fact that the commutator bracket of an algebra also has these properties. □

Remark 1.7. The situation that we will primarily be interested in is not quite the same as described above. Let A be a $\mathbb{C}$-algebra, $\mathbf{t}$ a central non-zero divisor and $\rho : \mathsf{A} \twoheadrightarrow A := \mathsf{A}/\mathbf{t} \cdot \mathsf{A}$ the quotient map. Assume that A is a finite module over $Z := Z(A)$, the centre of A. For each $z \in Z$, choose an arbitrary lift $\hat{z}$ of z in A. Write $A = \sum_{i=1}^{k} Z \cdot a_i$ and fix lifts $\hat{a}_i$ of a_i in A. Then the formula

$$\{z_1, z_2\} = \rho([\hat{z}_1, \hat{z}_2]/\mathbf{t}), \forall z_1, z_2 \in Z \tag{1.1}$$

defines a Poisson bracket on Z. Let us check that this is well-defined. First we should check that $\rho([\hat{z}_1, \hat{z}_2]/\mathbf{t}) \in Z$ for all $\hat{z}_1, \hat{z}_2 \in \rho^{-1}(Z)$. Let $a \in A$ and choose some lift $\hat{a}$ of a in A. Then

$$[a, \rho([\hat{z}_1, \hat{z}_2]/\mathbf{t})] = \rho([\hat{a}, [\hat{z}_1, \hat{z}_2]/\mathbf{t}]) = \rho([\hat{z}_2, [\hat{z}_1, \hat{a}]]/\mathbf{t}) + \rho([\hat{z}_1, [\hat{a}, \hat{z}_1]]/\mathbf{t}) = 0.$$

The fact that the definition of $\{-, -\}$ is independent of the choice of lifts follows from the fact that if $\hat{a}, \hat{z} \in \mathsf{A}$ such that $\rho(\hat{z}) \in Z$, then

$$\rho([\mathbf{t} \cdot \hat{a}, \hat{z}]/\mathbf{t}) = \rho([\hat{a}, \hat{z}]) = [\rho(\hat{a}), \rho(\hat{z})] = 0.$$

The fact that the bracket makes Z into a Lie algebra is a consequence of the fact that the commutator bracket of an algebra also has these properties. Let us just check that $\{-,-\}$ has the correct derivation property. Let $z_1, z_2, z_3 \in Z$ and choose lifts $\hat{z}_i$ of z_i in A. Since $\hat{z}_1 \cdot \hat{z}_2$ is a lift of $z_1 \cdot z_2$ in A, we have

$$\{z_1 \cdot z_2, z_3\} = \rho([\hat{z}_1 \cdot \hat{z}_2, \hat{z}_3]/\mathbf{t}) = \rho(\hat{z}_1([\hat{z}_2, \hat{z}_3]/\mathbf{t}) + ([\hat{z}_1, \hat{z}_3]/\mathbf{t}\hat{z}_2)) = z_1\{z_2, z_3\} + \{z_1, z_3\}z_2.$$

The assumption that A is a finite Z-module allows us to make A into a Poisson module for Z by defining

$$\{z, a_i\} = \rho([\hat{z}, \hat{a}_i]/\mathbf{t}), \, \forall z \in Z, i = 1, \ldots, k.$$

By repeating the arguments above, one can show that this operation is well-define, independent of the choice of lifts and satisfies the axioms of a Poisson module as given in [17, §4.1].

Assume now that $Z = \bigoplus_{i \in \mathbb{N}} Z_i$ is $\mathbb{N}$-graded. The Poisson bracket is said to be *graded* if there exists some $l \in \mathbb{Z}$ such that $\{-,-\} : Z_i \times Z_j \to Z_{i+j+l}$ for all $i, j \in \mathbb{N}$. Then l is said to be the degree of the bracket. Note that if $Z_0 = \mathbb{C}$ and Z is generated in degree one then every non-zero graded Poisson bracket on Z has degree ≥ -2.

1.4 Symplectic leaves for Poisson varieties

Given a Poisson variety $X = \mathrm{Maxspec}\,(Z)$ over $\mathbb{C}$ there are three stratifications of X that are induced by the Poisson structure. Two of these, the stratification by rank and stratification by symplectic leaves, we have met already in the context of Poisson manifolds (1.2). We will show that more or less the same definitions make sense in the algebraic setting. The third stratification is the stratification by Poisson cores as introduced in (1.3).

Definition 1.8. Let $X = \mathrm{Maxspec}\,(Z)$ be a Poisson variety and choose $\{z_1, \ldots, z_n\}$ a finite generating set for Z. For each $\mathfrak{m} \in X$, define $M(\mathfrak{m}) = (\{z_i, z_j\} + \mathfrak{m})_{i,j} \in \mathrm{Mat}_n(\mathbb{C})$. The *rank* of $\{-,-\}$ at $\mathfrak{m}$, denoted $\mathrm{rk}\,(\mathfrak{m})$, is defined to be the rank of the matrix $M(\mathfrak{m})$ (which is independent of the choice of generating set). For each $j \in \mathbb{N}$ define

$$X_j^o := \{\mathfrak{m} \in X \,|\, \mathrm{rk}\,(\mathfrak{m}) = j\}$$

and

$$X_j := \{\mathfrak{m} \in X \,|\, \mathrm{rk}\,(\mathfrak{m}) \leq j\}.$$

The rank stratification of X is defined to by $X = \bigsqcup_{j \in \mathbb{N}} X_j^o$.

The basic properties of the sets X_j^o and X_j which, in particular, show that the stratification is a well defined stratification of X into locally closed sets are summarized below.

Lemma 1.9 (Lemma 3.1,[17])**.** *For X a Poisson variety over $\mathbb{C}$ and X_j^o and X_j as above:*

1. *X_j is a closed subset of X, with*

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X,$$

where $n = \dim X$.

2. *X_j is a (not necessarily irreducible) Poisson subvariety of X.*

3. *The sets X_j^o are locally closed. If X_j^o is non-empty then $\overline{X_j^o}$ is the union of a certain number of irreducible components of X_j.*

It is useful to have the notion of symplectic leaves in the context of Poisson varieties. However, the stratification by symplectic leaves is not as simple to define in the algebraic setting since the notion of flows is analytic. We follow here the construction given in [17, §3.5]. Let us begin by assuming that X is a smooth variety. By changing the topology on X we can consider it as a complex analytic manifold. The ring of holomorphic functions on X will be denoted $\hat{Z}$. Since X is smooth, the Poisson bracket on X is encoded in a 2-vector $\Pi \in \Theta_X^{2,alg}(X) \subset \Theta_X^{2,hol}(X)$. Therefore the Poisson bracket on Z extends uniquely to a Poisson bracket on $\hat{Z}$. We define the symplectic leaves of X to be the symplectic leaves induced from this Poisson bracket on $\hat{Z}$. Now let us get back to the general situation. Set $I_0 := \sqrt{\{0\}}$, an ideal in Z. If J is any Poisson ideal in Z then it's radical $\sqrt{J}$ is a Poisson ideal [17, (2.4)], so I_0 is a Poisson ideal. Define inductively an ascending chain of ideals in Z by setting I_{j+1} to be the ideal of Z such that I_{j+1}/I_j is the radical ideal defining the singular locus of $\mathrm{Maxspec}\, Z/I_j$. This stabilizes to $I_m = Z$. By [82, Corollary 2.4] each I_j is a semi-prime Poisson ideal in Z. Let us write $\mathcal{L}_j$ for $\mathrm{Maxspec}\, Z/I_j$. The smooth locus, $(\mathcal{L}_j)_{sm}$ of $\mathcal{L}_j$ can be considered a complex analytic Poisson manifold. Therefore it has a stratification by symplectic leaves. Hence for each $j = 0, \ldots, m$, there is an indexing set $\mathcal{I}_j$ such that

$$(\mathcal{L}_j)_{sm} = \bigsqcup_{k \in \mathcal{I}_j} \mathcal{L}_{j,k}$$

is the stratification of $(\mathcal{L}_i)_{sm}$ by symplectic leaves. This allows us to define the stratification of X by symplectic leaves as

$$X = \bigsqcup_{0 \le j \le m;\, k \in \mathcal{I}_j} \mathcal{L}_{j,k}.$$

In general, the sets $\mathcal{L}_{j,k}$ are not locally closed in the Zariski topology. However, [17, Lemma 3.5] says that the closure $\overline{\mathcal{L}_{j,k}}$ of $\mathcal{L}_{j,k}$ is defined by a prime Poisson ideal, $K_{j,k}$ say, and $K_{j,k}$ is the Poisson core $\mathcal{C}(\mathfrak{m})$ of every $\mathfrak{m} \in \mathcal{L}_{j,k}$. Often the symplectic leaves that arise in representation theoretic situations are particular nice - the sets $\mathcal{L}_{j,k}$ are actually locally closed in the Zariski topology. Again following [17], we will say that the Poisson bracket is *algebraic* if every symplectic leaf $\mathcal{L}_{j,k}$ of X is a locally closed set.

Generally, the three stratifications defined on X will not agree. However when $\{-,-\}$ is algebraic and the number of leaves is finite, the three stratifications are actually (more or less) the same. To be precise:

Proposition 1.10 (Proposition 3.7, [17]). *Let X be a Poisson variety over $\mathbb{C}$ such that the number of symplectic leaves is finite. Then:*

- *The Poisson bracket is algebraic.*
- *Let $\mathfrak{m} \in X$ with $rk(\mathfrak{m}) = j$. The following subsets of X coincide:*
  1. *the symplectic leaf $\mathcal{L}(\mathfrak{m})$ containing $\mathfrak{m}$,*
  2. *$\mathcal{Q}(\mathfrak{m})$,*
  3. *the irreducible component of X_j^o containing $\mathfrak{m}$,*
  4. *the smooth locus of the irreducible component of X_j containing $\mathfrak{m}$.*

A simple example where one gets an infinite number of symplectic leaves and the above example fails is given by $Z = \mathbb{C}[x, y]$ with bracket $\{x, y\} = x$. The symplectic leaves of $\mathbb{C}^2$ are $\mathbb{C}^2 \backslash V(x)$ and $\{(0, \alpha)\}_{\alpha \in \mathbb{C}}$ but there are only two strata in the stratification by symplectic cores: $\mathbb{C}^2 \backslash V(x)$ and $V(x)$. In chapter 4 we will introduce a class of Poisson varieties called symplectic varieties. Kaledin [65, Theorem 2.3] has shown that the Poisson bracket on a symplectic variety is algebraic.

Example 1.11. We return to the example (1.4) of the Poisson manifold $\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of a connected complex Lie group G. Let us make the additional assumption that G is a semi-simple algebraic group and we consider $\mathfrak{g}^*$ as a Poisson variety. Since the symplectic leaves in $\mathfrak{g}^*$ are the coadjoint orbits, the functions in $\mathbb{C}[\mathfrak{g}^*]^G$ are constant on leaves. A theorem by Chevalley, see for instance [58, §23.1], says that $\mathbb{C}[\mathfrak{g}^*]^G \simeq \mathbb{C}[\mathfrak{h}^*]^W$ where $\mathfrak{h}^*$ is the dual of a Cartan subalgebra of $\mathfrak{g}$ and W the associated Weyl group. This shows that every fiber of the quotient map $\pi : \mathfrak{g}^* \twoheadrightarrow \mathfrak{h}^*/W$ is a union of coadjoint orbits. In particular, there are an infinite number of orbits in $\mathfrak{g}^*$. However it was shown by Steinberg [95], see also [89], that each fiber $\pi^{-1}(p)$ is a finite union of orbits. Therefore Proposition 1.10 implies that the Poisson bracket on the Poisson variety $\pi^{-1}(p)$ is algebraic. The fiber $\mathcal{N} := \pi^{-1}(0)$ is the most interesting and is the set of nilpotent elements in $\mathfrak{g}^*$. It is called the *nilpotent cone*. If $\mathbb{C}[\mathfrak{g}^*]$ is graded so that the degree of $\mathfrak{g}$ is one then the Poisson bracket on $\mathbb{C}[\mathfrak{g}^*]$ is graded of degree -1. The ideal defining $\mathcal{N}$ in $\mathfrak{g}^*$ is also graded therefore the Poisson bracket on $\mathcal{N}$ has degree -1.

1.5 Complex reflection groups

Let $\mathfrak{h}$ be a finite dimensional vector space. A subgroup $W \subset GL(\mathfrak{h})$ is called a *complex reflection group* if it is generated by the *pseudo-reflections* that it contains. Here a pseudo-reflection is an element $g \in W$ such that rank $(1 - g) = 1$. One should think of a complex reflection group as a pair $(W, \mathfrak{h})$ but we will simply write W with the implicit assumption that a reflection representation has been fixed for W. We will only consider finite complex reflection groups. The complex reflection group is said to be irreducible if the reflection representation is irreducible as a W-module. The rank of W is then defined to be the dimension of $\mathfrak{h}$. The irreducible complex reflection groups are divided into two classes, the primitive complex reflection groups and the imprimitive complex reflection groups. They have been classified by Shephard and Todd in [88]. There are thirty four primitive complex reflection groups, which in the classification of [88] are labeled $G_4, \dots, G_{37}$. They are also commonly known as the exceptional complex reflection groups. The imprimitive complex reflection groups belong to one infinite family $G(m, d, n)$ where $m, d, n \in \mathbb{N}$ and d divides m. Let S_n be the symmetric group on n elements, considered as the group of all $n \times n$ permutation matrices, and let $A(m, d, n)$ be the group of all diagonal matrices where the diagonal entries are powers of a certain (fixed) m^{th} root of unity and the determinant of each matrix is a $(m/d)^{th}$ root of unity. The group S_n normalizes $A(m, d, n)$ and $G(m, d, n)$ is defined to be the semi-direct product of $A(m, d, n)$ by S_n.

Chevalley, [21] showed that the ring of invariants $\mathbb{C}[\mathfrak{h}]^W$ is especially well behaved. The converse to Chevalley's Theorem was proved by Shephard and Todd in their classification paper [88]. It gives another characterization of complex reflection groups.

Theorem 1.12 ([21] and [88]). *Let $W \subset GL(\mathfrak{h})$ be a finite group. The following are equivalent:*

- *W is a complex reflection group.*
- *The ring of invariants $\mathbb{C}[\mathfrak{h}]^W$ is generated by* $\dim \mathfrak{h}$ *algebraically independent, homogeneous elements.*

Now assume W is any finite subgroup of $GL(\mathfrak{h})$ and let us denote by $\mathbb{C}[\mathfrak{h}]^W_+$ the ideal of all function with constant term zero (the "augmentation ideal"). The ring of co-invariants for W is defined to be the finite dimensional quotient algebra $\mathbb{C}[\mathfrak{h}]^{coW} := \mathbb{C}[\mathfrak{h}]/\langle \mathbb{C}[\mathfrak{h}]^W_+ \rangle$. In general this is a very complicated ring to understand, for instance when S_n acts diagonally on $\mathbb{C}^{2n}$ the description of $\mathbb{C}[\mathbb{C}^{2n}]^{coS_n}$ as a graded S_n-module was only recently given by Haiman and is a consequence of his proof of the $n!$-conjecture, see [55]. However, when W is a complex reflection group, Chevalley in the same paper [21] gave a description of $\mathbb{C}[\mathfrak{h}]^{coW}$ as a W-module.

Proposition 1.13 ([21]). *Let W be a complex reflection group. Then as a W-module the ring of co-invariants $\mathbb{C}[\mathfrak{h}]^{coW}$ is isomorphic to the regular representation. In particular,* $\dim \mathbb{C}[\mathfrak{h}]^{coW} = |W|$.

1.6 The skew group ring

Let V be a finite dimensional vector space over $\mathbb{C}$ and fix $G \subset GL(V)$ a finite group. The group G acts on the polynomial ring $\mathbb{C}[V]$ of functions on V. We can form the *skew group ring* $\mathbb{C}[V] \rtimes G$, which as a vector space is $\mathbb{C}[V] \otimes \mathbb{C}G$ and where multiplication is given by

$$(f_1 \otimes g_1) \cdot (f_2 \otimes g_2) = f_1 g_1(f_2) \otimes g_1 g_2 \quad \forall\, f_i \in \mathbb{C}[V], g_i \in G.$$

The ring $\mathbb{C}[V] \rtimes G$ enjoys many very nice properties, some of which we list here.

Proposition 1.14. *The skew group ring has the following properties:*

1. *It is a prime, Noetherian ring.*
2. *The global dimension of $\mathbb{C}[V] \rtimes G$ equals* $\dim V$.

3. *The centre of* $\mathbb{C}[V] \rtimes G$ *is* $\mathbb{C}[V]^G$.

Proof. Since $\mathbb{C}[V]$ is an integral domain and G acts faithfully on $\mathbb{C}[V]$, [81, Corollary 12.6] says that $\mathbb{C}[V] \rtimes G$ is a prime ring. By [74, Theorem 1.5.12], the fact that $\mathbb{C}[V]$ is Noetherian implies that $\mathbb{C}[V] \rtimes G$ is Noetherian. Hilbert's Syzygies Theorem, [85, Theorem 8.37], says that $\mathrm{gl.dim}\,(\mathbb{C}[V]) = \dim V$. By [74, Theorem 1.5.12], $\mathrm{gl.dim}\,(\mathbb{C}[V] \rtimes G) = \mathrm{gl.dim}\,(\mathbb{C}[V])$. A direct calculation shows that the centre of $\mathbb{C}[V] \rtimes G$ is $\mathbb{C}[V]^G$. □

The Hilbert-Noether Theorem, see [8, Theorem 1.3.1], says that the ring of invariants $\mathbb{C}[V]^G$ is finitely generated. Since $\mathbb{C}[V]^G \subset \mathbb{C}[V]$, it is a domain and hence the ring of functions on an affine variety V/G. Since G is finite and we are working over $\mathbb{C}$, the closed points of V/G can be identified with the G-orbit in V, as shown in [8, Theorem 1.4.4]. Let L be a simple $\mathbb{C}[V] \rtimes G$-module. The general theory described in (3.1) shows that L is finite dimensional. Then Schur's Lemma says that the centre of $\mathbb{C}[V] \rtimes G$ acts on L as scalars. Therefore, considered as a $\mathbb{C}[V]^G$-module, L is supported at some point $p \in V/G$. If M is a $\mathbb{C}[V] \rtimes G$-module and $g \in GL(V)$ we denote by ${}^gM$ the $\mathbb{C}[V] \rtimes (g \cdot G \cdot g^{-1})$-module that as a vector space equals M but with action

$$f \cdot_g m = (g^{-1} f g) \cdot m \quad \forall f \in \mathbb{C}[V] \rtimes (g \cdot G \cdot g^{-1}).$$

The simple modules for the skew group ring are described as follows:

Proposition 1.15. *Choose a point* $p \in V/G$ *and* $q \in V$ *such that* $G \cdot q = p$. *Let* $H = \mathit{Stab}_G(q)$. *Then the map*

$$\lambda \mapsto \mathbb{C}[V] \rtimes G \otimes_{\mathbb{C}[V] \rtimes H} \lambda$$

defines a bijection between the set $\mathsf{Irr}(H)$ *of non-isomorphic simple* H*-modules and the set of non-isomorphic simple* $\mathbb{C}[V] \rtimes G$*-modules supported at the point* p*, where* $\mathbb{C}[V]$ *acts on* λ *as* $f \cdot v = f(q)v$ *for* $f \in \mathbb{C}[V]$ *and* $v \in \lambda$. *In particular, a generic simple* $\mathbb{C}[V] \rtimes G$*-module is isomorphic to the regular representation as a* G*-module.*

Proof. Fix $C = \{g_1, \ldots, g_n\}$ to be a set of coset representatives of H in G. Let L be a simple $\mathbb{C}[V] \rtimes G$-module supported at p. Since L is finite dimensional, the support of L as a $\mathbb{C}[V]$-module will be a finite number of (possibly non-reduced) points. Let q_1 be one of these points and L_1 the direct summand of L supported at q_1. Then $\bigoplus_{g \in C} {}^gL_1$ is a $\mathbb{C}[V] \rtimes G$-submodule of L, whose support is precisely $G \cdot q_1 = p$. Therefore $\bigoplus_{g \in C} {}^gL_1 = L$ and we may assume without loss of generality that $q = q_1$. The subalgebra $\mathbb{C}[V] \rtimes H$ acts on L_1 and we can choose a simple $\mathbb{C}[V] \rtimes H$-submodule M of L_1. Then $\bigoplus_{g \in C} {}^gM$ is a $\mathbb{C}[V] \rtimes G$-submodule of L hence $M = L_1$ is irreducible. Therefore we may now assume without loss of generality that $G = H$ is the stabilizer of $q = 0$ in V. This means that L is a simple module for the finite dimensional algebra $\mathbb{C}[V]^{coG} \rtimes G$. Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{C}[V]$ defining 0. The image of $\mathfrak{m}$ in $\mathbb{C}[V]^{coG}$ is a nilpotent ideal. Let $L_0 \subset L$ be the (non-empty) subspace of elements v such that $\mathfrak{m} \cdot v = 0$. Then one can check that L_0 is a $\mathbb{C}[V]^{coG} \rtimes G$-submodule of L, hence $L = L_0$. Take $\lambda \subset L_0 = L$ to be a simple G-module. It will be a simple $\mathbb{C}[V]^{coG} \rtimes G$-submodule of L hence $L = \lambda$. Running this argument in reverse shows that the $\mathbb{C}[V] \rtimes G$-modules $\mathbb{C}[V] \rtimes G \otimes_{\mathbb{C}[V] \rtimes H} \lambda$ are simple and $\mathbb{C}[V] \rtimes G \otimes_{\mathbb{C}[V] \rtimes H} \lambda_1 \simeq \mathbb{C}[V] \rtimes G \otimes_{\mathbb{C}[V] \rtimes H} \lambda_2$ if and only if $\lambda_1 \simeq \lambda_2$. □

Let us now assume that (V, ω) is a symplectic vector space. An element $g \in Sp(V)$ is said to be a *symplectic reflection* if $\dim\{v \in V | g \cdot v = v\} = \dim V - 2$. A group $G \subset Sp(V)$ that is generated by symplectic reflections is called a *symplectic reflection group.* Following [36], we say that (V, ω, G) is an *indecomposable triple* if there is no ω-orthogonal decomposition $V = V_1 \oplus V_2$ into proper G-stable subspaces V_1 and V_2.

Proposition 1.16. *Let G be a symplectic reflection group. Then the skew group ring $\mathbb{C}[V] \rtimes G$ is a maximal order in its simple ring of fractions and its centre $\mathbb{C}[V]^G$ is integrally closed.*

Proof. Proposition 1.14 says that $\mathbb{C}[V] \rtimes G$ is a prime Noetherian ring. Therefore Goldie's Theorem [47, Theorem 6.15] says that it is contained in its simple Artinian ring of fractions. The (Zariski closed) set of points in V where the group G does not act freely has codimension at least two. The result [71, Theorem 4.6] says that this implies that the skew group ring $\mathbb{C}[V] \rtimes W$ is a maximal order. The centre of a maximal order is integrally closed, see [74, Proposition 5.1.10]. □

When V is a symplectic vector space and G a group acting by symplectomorphisms the natural Poisson bracket on V descends to a Poisson bracket on V/G making it a Poisson variety.

Lemma 1.17. *Let (V, ω) be a symplectic vector space and $G \subset Sp(V)$ a finite group. Then the Poisson bracket on $\mathbb{C}[V]$ restricts to a Poisson bracket on the subalgebra $\mathbb{C}[V]^G$.*

Proof. Since $\omega(g \cdot u, g \cdot v) = \omega(u, v)$ for all $u, v \in V$, we see from the definition of the bracket on $\mathbb{C}[V]$ that $\{g \cdot \lambda, g \cdot \mu\} = \{\lambda, \mu\} \in \mathbb{C}$ for all $\lambda, \mu \in V^* \subset \mathbb{C}[V]$. The Leibniz rule then implies that $\{g \cdot f_1, g \cdot f_2\} = g \cdot \{f_1, f_2\}$. This shows that $\{-,-\}$ restricts to a bracket $\mathbb{C}[V]^G \times \mathbb{C}[V]^G \to \mathbb{C}[V]^G$ as required. □

The polynomial ring $\mathbb{C}[V]$ is $\mathbb{N}$-graded by putting V^* in degree one. Then the Poisson bracket on $\mathbb{C}[V]$ has degree -2. This means that the bracket on V/G is also of degree -2.

Lemma 1.18. *Let (V, ω, G) be an indecomposable triple.*

1. *Either V is a simple G-module or $V = U \oplus U^*$ with U a simple G-module and U, U^* Lagrangian with respect to ω.*

2. *If $V = U \oplus U^*$ then G acts on U as a complex reflection group.*

3. *The space $(\Lambda^2 V^*)^G$ is one-dimensional.*

Proof. Let $V = U_1 \oplus \cdots \oplus U_k$ be a decomposition of V into simple G-modules. Let us assume that $V \neq U_1$. Then I claim that U_1 is an isotropic subspace of V. The space U_1 cannot be a symplectic subspace of V because of indecomposability. Therefore the subspace Ker $\omega_{|\,U_1}$ is a non-zero submodule of U_1. This implies that $U_1 =$ Ker $\omega_{|\,U_1}$, confirming the claim. Now take $0 \neq u \in U_1$. We can find some $j \neq 1$ and $v \in U_j$ such that $\omega(u, v) \neq 0$. Since U_1 is simple the elements $g \cdot u$, $g \in G$, span U_1. Then $\omega(g \cdot u, g \cdot v) = \omega(u, v) \neq 0$ implies that $U_1 \oplus U_j$ is a symplectic subspace of V. Therefore indecomposability implies that $V = U_1 \oplus U_j$. The symplectic form induces a G-module isomorphism $\omega : U_j \to U_1^*$. A symplectic reflection in G must act on U as a pseudo-reflection. Since these elements generate G, it acts on U as a complex reflection group.

Now we will show that $\dim(\Lambda^2 V^*)^G = 1$. Since $\omega \in (\Lambda^2 V^*)^G$, the dimension of this space is at least one. Choose $0 \neq \nu \in (\Lambda^2 V^*)^G$ and decompose $V = \operatorname{Ker} \nu \oplus V'$, where V' is some G-module. If V' is a proper submodule of V then we must have $V = U \oplus U^*$ and $V' \simeq U$. However (V', ν, G) is a symplectic reflection group and $G \subset Sp(V')$ contradicts the fact that G acts on V' as a complex reflection group. Therefore ν must be non-degenerate and $V' = V$. Take $s \in G$ to be a symplectic reflection and set $H = \langle s \rangle$. We can decompose $V = \operatorname{Im}(1-s) \oplus \operatorname{Ker}(1-s)$ into symplectic vector spaces (with respect to both ω and ν). Let $V_0 := \operatorname{Im}(1-s)$. It is a two-dimensional vector space. The restrictions ν_0 and ω_0 of ν and ω to V_0 are non-zero elements in the one-dimensional space $\Lambda^2 V_0^* = (\Lambda^2 V_0^*)^H$. After rescaling ν if necessary, we may assume $\nu_0 = \omega_0$. If $\nu \neq \omega$ then $\nu - \omega$ is a non-zero but degenerate element in $(\Lambda^2 V^*)^G$, which we have shown is not possible. Therefore $\nu = \omega$. □

Lemma 1.19 (Lemma 2.23, [36])**.** *Let (G, V, ω) be an indecomposable triple. Then every non-zero graded Poisson bracket on $\mathbb{C}[V]^G$ is proportional to $\{-,-\}_\omega$ and hence has degree -2.*

Proof. Let V_{reg} be the set of point where G acts freely. A Poisson bracket on $\mathbb{C}[V]^G$ restricts to a Poisson bracket on $\mathcal{O}_{V_{\text{reg}}/G}$. The fact that V_{reg}/G is smooth means that $\Theta^2_{V_{\text{reg}}/G}$ is a vector bundle on V_{reg}/G and the Poisson structure on V_{reg}/G corresponds to $\Pi \in \Theta^2_{V_{\text{reg}}/G}$. Since $\Theta^2_{V_{\text{reg}}/G} = \pi_*(\Theta^2_{V_{\text{reg}}})^G$, Π defines a G-invariant Poisson bracket on V_{reg}. The fact that $G \subset Sp(V)$ implies that the complement of V_{reg} in V has codimension at least two. Since V is smooth, and hence normal, Π extends to a regular vector field on the whole of V, see [34, (11.2)]. Hence Π corresponds to a nonzero element in $(\Lambda^2 V)^G$. As explained in Lemma 1.18, this means that the bracket must be proportional to ω. □

The symplectic leaves of V/G have been described by Brown and Gordon [17, §7.4]. We recall here their description. Let H be a subgroup of G and write V^H for the subspace of V consisting of all elements $v \in V$ such that H is contained in the stabilizer of v. We will denote by $V^H_{\text{reg}} := (V^H)_{\text{reg}}$ the set of all elements v in V^H such that H equals the stabilizer of v. Note that this set may be empty. The sets V^H_{reg} define a stratification of V by locally closed sets:

$$V = \bigsqcup_{H \leq G} V^H_{\text{reg}}.$$

Set $\mathcal{L}_H := \pi(V^H_{\text{reg}})$ where $\pi : V \twoheadrightarrow V/G$ is the quotient map. The set $\mathcal{L}_H$ is locally closed in V/G and depends only on the conjugacy class of H in G. This gives a stratification of V/G:

$$V/G = \bigsqcup_{H \leq G} \mathcal{L}_H.$$

Proposition 1.20 (Proposition 7.4, [17])**.** *The symplectic leaves of V/G are the sets $\mathcal{L}_H$ as H runs through all conjugacy classes of subgroups of G such that V^H_{reg} is non-empty.*

We will briefly return to the question of labeling the symplectic leaves of V/G in Corollary 5.29.

1.7 Remarks

1. Our main reference on Poisson manifolds is the book [33], the article by Weinstein [101] also contains a wealth of information. The results on Poisson algebras were taken almost entirely from the paper [17].

2. The facts about skew-group rings that we stated were taken from [74], many more of their properties are described in that book. Our main source on invariant rings was the book [8], which also explores their properties in prime characteristic.

Chapter 2

Rational Cherednik algebras

2.1 Symplectic reflection algebras

Rational Cherednik algebras form a subset of the slightly more general class of algebras called *Symplectic reflection algebras*. Since many of the results in this chapter can be stated and proved just as easily for this larger class of algebras it seems natural to introduce them here. Let (V, ω) be a symplectic vector space and fix G a finite subgroup of $Sp(V)$ such that (V, ω, G) is an indecomposable triple. The properties of $\mathbb{C}[V] \rtimes G$ can be quite complicated. One way to try and overcome this problem is to try and construct a family of algebras such that $\mathbb{C}[V] \rtimes G$ occurs a special fiber of this family. Then it is reasonable to expect that a generic algebra in this family should be easier to study and this in turn would allow us to deduce something about our original algebra $\mathbb{C}[V] \rtimes G$. This is one of the many ideas behind Etingof and Ginzburg's construction of symplectic reflection algebras.

Let TV^* denote the tensor algebra on V^*. The group G acts on TV^* and we can form the corresponding skew group ring $TV^* \rtimes G$. It is a graded algebra if we put V^* in degree one and $\mathbb{C}G$ in degree zero. The skew group ring $\mathbb{C}[V] \rtimes G$ is the graded quotient of $TV^* \rtimes G$ by the two-sided ideal generated by the quadratic elements $v \otimes w - w \otimes v$, $v, w \in V^*$. In order to construct deformations of this ring, let us fix $\kappa : V^* \times V^* \to \mathbb{C}G$ to be a skew-symmetric, $\mathbb{C}$-bilinear pairing. Then define

$$H_\kappa := TV^* \rtimes G / \langle v \otimes w - w \otimes v - \kappa(v, w) \,|\, v, w \in V^* \rangle.$$

It is an associative algebra and taking $\kappa = 0$ recovers the skew group ring $\mathbb{C}[V] \rtimes G$. When $\kappa \neq 0$ the quotient ideal is no longer graded therefore H_κ is not naturally graded. However it inherits an $\mathbb{N}$-filtration $\mathcal{F}_\bullet(H_\kappa)$ and we can form the associated graded algebra $\mathsf{gr}(H_\kappa) := \bigoplus_{i \geq 0} \mathcal{F}_i(H_\kappa)/\mathcal{F}_{i-1}(H_\kappa)$. If $v, w \in \mathsf{gr}(H_\kappa)$ lie in the image of V^* under the quotient map then it is clear from the defining relations of H_κ that $[v, w] = 0$. Therefore the natural map $V^* \to \mathsf{gr}(H_\kappa)$ extends to an algebra morphism $\mathbb{C}[V] \rtimes G \to \mathsf{gr}(H_\kappa)$. Following Etingof and Ginzburg, we say that the *Poincaré-Birkhoff-Witt (PBW) property* holds for H_κ if this morphism is an isomorphism. Let us denote by $\mathcal{S}(G)$ the set of symplectic reflections in G. For each $s \in \mathcal{S}(G)$, the spaces $\mathrm{Im}\,(1 - s)$ and $\mathrm{Ker}\,(1 - s)$ are symplectic subspaces of

V with $V = \mathrm{Im}\,(1-s) \oplus \mathrm{Ker}\,(1-s)$ and $\dim \mathrm{Im}\,(1-s) = 2$. We denote by ω_s the 2-form on V whose restriction to $\mathrm{Im}\,(1-s)$ is ω and whose restriction to $\mathrm{Ker}\,(1-s)$ is zero. The crucial result by Etingof and Ginzburg, on which the whole of the theory of symplectic reflection algebras is built, is the "PBW Theorem".

Theorem 2.1 (Theorem 1.3, [36]). *Let (V, ω, G) be an indecomposable triple. Then the algebra H_κ has the PBW property if and only if there exists a constant $t \in \mathbb{C}$ and function $\mathbf{c} : \mathcal{S}(G) \to \mathbb{C}$, constant on conjugacy classes, such that the pairing κ has the form*

$$\kappa(v, w) = t \cdot \omega(v, w) \cdot 1 + \sum_{s \in \mathcal{S}(G)} \mathbf{c}(s) \cdot \omega_s(v, w) \cdot s, \quad \forall\, v, w \in V^*. \tag{2.1}$$

The proof of this theorem is an application of a general result by Braverman and Gaitsgory [11]. If I is a two-sided ideal of TV^* generated by a space U of (not necessarily homogeneous) elements of degree at most two then [11, Theorem 0.5] gives necessary and sufficient conditions on U so that the quotient TV^*/I has the PBW property. From now on, we will always assume that κ has the form described in the above theorem and write $H_\kappa = H_{t,\mathbf{c}}$. Let $\mathbf{e} = \frac{1}{|G|}\sum_{g \in G} g$ denote the trivial idempotent in $\mathbb{C}G$. The subalgebra $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e} \subset H_{t,\mathbf{c}}$ is called the *spherical subalgebra* of $H_{t,\mathbf{c}}$. Being a subalgebra it inherits a filtration from $H_{t,\mathbf{c}}$. It is a consequence of the PBW theorem that $\mathsf{gr}(\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}) \simeq \mathbb{C}[V]^G$.

2.2 The centre of $H_{t,\mathbf{c}}(G)$

In this section we establish the fundamental result by Etingof and Ginzburg which says that the spherical subalgebra of the symplectic reflection algebra is commutative if and only if $t = 0$. This, together with a result of Brown and Gordon, implies that the symplectic reflection algebra is a finite module over its centre if and only if $t = 0$. The proof is a very clever argument using the multiplication in the spherical subalgebra of $H_{t,\mathbf{c}}$ to define a Poisson bracket on V/G. The key fact is Lemma 1.19 - there is, up to rescaling, a unique non-zero Poisson bracket on V/G. To begin with let us assume that $(t, \mathbf{c})$ is chosen such that $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ is commutative. The space $H_{t,\mathbf{c}}\mathbf{e}$ is a finite module over $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$, or equivalently a coherent sheaf on $\mathrm{Spec}\,(\mathbf{e}H_{t,\mathbf{c}}\mathbf{e})$. We call it the *Etingof-Ginzburg sheaf* and denote it $\mathcal{R}[G]$. Note that $H_{t,\mathbf{c}}$ and, in particular, G act on $\mathcal{R}[G]$ on the left.

Lemma 2.2 (Lemma 2.24, [36]). *Let p be a generic point of Spec $(\mathbf{e}H_{t,\mathbf{c}}\mathbf{e})$, then the fiber $\mathcal{R}[G](p)$ is isomorphic to the regular representation as a G-module.*

Proof. Let H be the quotient of $(TV^* \rtimes G) \otimes \mathbb{C}[r]$ by the ideal generated by $\{v \otimes w - w \otimes v - r \cdot \kappa(v, w) \mid v, w \in V^*\}$, where $\kappa(v, w)$ is defined in (2.1). Since the algebras $H_{\nu \cdot t, \nu \cdot \mathbf{c}}$ are isomorphic for all $\nu \in \mathbb{C}^*$, the $\mathbb{C}[r]$-algebra eHe is a commutative algebra. We denote by the same symbol $\mathcal{R}[G]$ the Etingof-Ginzburg sheaf on Spec(eHe). The PBW theorem says that H is flat over $\mathbb{C}[r]$. Since eHe and He are direct summands of H as $\mathbb{C}[r]$-modules, eHe and $\mathcal{R}[G]$ are flat $\mathbb{C}[r]$-modules. Being a G-equivariant sheaf, $\mathcal{R}$ decomposes as

$$\mathcal{R} = \bigoplus_{\lambda \in \mathsf{Irr}(G)} \lambda \otimes \mathcal{R}_\lambda.$$

Thus we wish to show that $\mathrm{rk}_{\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}}(\mathcal{R}_\lambda/(r-1)\cdot\mathcal{R}_\lambda) = \dim\lambda$ as a coherent sheaf on $\mathrm{Spec}\,(\mathbf{e}H_{t,\mathbf{c}}\mathbf{e})$. Proposition 1.15 shows that $\mathrm{rk}_{\mathbf{e}H_{0,0}\mathbf{e}}(\mathcal{R}_\lambda/r\cdot\mathcal{R}_\lambda) = \dim\lambda$. The required equality follows from the following claim, whose proof can be found in [36, page 266].

Claim Let A be a commutative $\mathbb{C}[r]$-algebra such that A is a domain, a flat $\mathbb{C}[r]$-module, finitely generated as a $\mathbb{C}[r]$-algebra and $A/(r-q)\cdot A$ is a domain for all $q \in \mathbb{C}$. If M is a finitely generated A-module, flat over $\mathbb{C}[r]$, then

$$\mathrm{rk}_A(M) = \mathrm{rk}_{A_q}(M/(r-q)\cdot M), \quad \forall\, q \in \mathbb{C},$$

where $A_q := A/(r-q)\cdot A$, $\mathrm{rk}_A(M) := \dim_{Q(A)}(Q(A)\otimes_A M)$ and $Q(A)$ the field of fractions of A. $\square$

Theorem 2.3 (Theorem 1.6, [36])**.** *The algebra $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ is commutative if and only if $t=0$.*

Proof. First we show, following the proof of [36, Theorem 1.6] that $t=0$ implies that $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ is commutative. As in the proof of Lemma 2.2, let eHe denote the $\mathbb{C}[r]$-algebra defined by $(t,\mathbf{c})$. It will be commutative if and only if $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ is commutative. For every $u,v \in \mathbb{C}[V]^G = \mathsf{eHe}/r\cdot\mathsf{eHe}$ we choose lifts $\tilde{u},\tilde{v} \in \mathsf{eHe}$. Since $\mathbb{C}[V]^G$ is commutative, there exists some $a \geq 1$ such that $[\tilde{u},\tilde{v}] \in r^a\cdot\mathsf{eHe}$. Let $\mathbf{m} \in \mathbb{N}\cup\{\infty\}$ be the largest integer such that $[\tilde{u},\tilde{v}] \in r^{\mathbf{m}}\cdot\mathsf{eHe}$ for all $u,v \in \mathbb{C}[V]^G$. Then

$$u,v \mapsto ([\tilde{u},\tilde{v}]/r^{\mathbf{m}}) \bmod r\cdot\mathsf{eHe}$$

defines a Poisson bracket $\{-,-\}_{(t,\mathbf{c})}$ on $\mathbb{C}[V]^G$ that is independent of the choice of lifts. The algebra eHe is $\mathbb{N}$-filtered, $\mathcal{F}_\bullet(\mathsf{eHe})$, if we put V in degree one and r in degree zero. The spaces $\mathcal{F}_i(\mathsf{eHe})$ are finite rank, flat $\mathbb{C}[r]$-modules. Therefore they are free $\mathbb{C}[r]$-modules. Let $f \in \bigcap_{i\geq 0} r^i\cdot\mathsf{eHe}$. Then there exists some j such that $f \in \mathcal{F}_j(\mathsf{eHe})$. However $\bigcap_{i\geq 0} r^i\cdot\mathcal{F}_j(\mathsf{eHe}) = 0$ implies that $f=0$ hence $\bigcap_{i\geq 0} r^i\cdot\mathsf{eHe} = 0$. If $\mathbf{m}=\infty$ then $\bigcap_{i\geq 0} r^i\cdot\mathsf{eHe} = 0$ implies that $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ is commutative. If we now assign $\deg(V)=1$ and $\deg(r)=2$ then the relations (2.1) show that eHe is an $\mathbb{N}$-graded algebra and it is clear from the construction of the bracket on $\mathbb{C}[V]^G$ that it is graded of degree $-2\mathbf{m}$. Therefore Lemma 1.19 says that $\mathbf{m}=1$ or ∞. Moreover, there exists a function $f : \mathbb{C}\times\mathcal{S}(W) \to \mathbb{C}$ such that $\{-,-\}_{(t,\mathbf{c})} = f(t,\mathbf{c})\cdot\{-,-\}_\omega$, where $\{-,-\}_\omega$ is the standard bracket on V/G as defined in Lemma 1.17. The algebra $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ will be commutative if and only if the bracket $\{-,-\}_{(t,\mathbf{c})}$ is zero i.e if and only if $f(t,\mathbf{c})=0$. Therefore the statement of the Theorem will follow if we can show that $f=\lambda t$ for some $\lambda \in \mathbb{C}^*$.

Let us now think of t and $\mathbf{c}$ as variables so that eHe is a $\mathbb{C}[r,t,\mathbf{c}]$-module (we use $\mathbf{c}$ here to denote an n-tuple of variables $c_1,c_2,\dots$). We begin by showing that $f(t,\mathbf{c})$ is a linear function. If we assign a new grading to eHe by saying that $\deg(V)$ is still one but $\deg(r)=0$ and $\deg(t)=\deg(\mathbf{c})=2$ then the defining relations of $H_{r\cdot t,r\cdot\mathbf{c}}$ are homogeneous and hence $H_{r\cdot t,r\cdot\mathbf{c}}$ and eHe are graded algebras. Again, Lemma 1.19 implies that if $f(t,\mathbf{c}) \neq 0$ then $\deg(f)=2$. This implies that f is linear. Now we just need to show that t divides f. Assume that we have chosen $(t,\mathbf{c})$ such that $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ is commutative. Fix L a generic simple $H_{t,\mathbf{c}}$-module with $\rho : H_{t,\mathbf{c}} \to \mathrm{End}_{\mathbb{C}}(L)$ the structure morphism. Then Lemma 2.2 says that L is isomorphic to the regular representation hence $tr(\rho(g)) = 0$ for all $1 \neq g \in G$. Applying traces to the defining relation

$$[\rho(v), \rho(w)] = t \cdot \omega(v, w) \cdot \rho(1) + \sum_{s \in \mathcal{S}(G)} \mathbf{c}(s) \cdot \omega_s(v, w) \cdot \rho(s) \quad \forall\, v, w \in V$$

and using the non-degeneracy of ω shows that $0 = |G| \cdot t$. Hence $f(t, \mathbf{c}) = \lambda t$ for some λ. Taking $\mathbf{c} = 0$ shows that $\lambda = 1$. $\square$

The following result shows that one can recover $H_{t,\mathbf{c}}(G)$ from knowing $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ and the Etingof-Ginzburg sheaf $H_{t,\mathbf{c}}\mathbf{e}$.

Theorem 2.4. *For all $(t, \mathbf{c})$, left multiplication by $H_{t,\mathbf{c}}(G)$ defines an isomorphism*

$$\psi \,:\, H_{t,\mathbf{c}}(G) \to End_{\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}}(H_{t,\mathbf{c}}\mathbf{e})$$

We wish to understand the centre of the symplectic reflection algebra. The Satake isomorphism allows us to relate this to the spherical subalgebra so that we can make use of the above results. The isomorphism is only stated for $t = 0$ in [36] but, as noted in [17, §7.2], makes sense for all t.

Theorem 2.5 (Theorem 3.1 (Satake isomorphism), [36])**.** *The map $z \mapsto z \cdot \mathbf{e}$ defines an algebra isomorphism $Z(H_{t,\mathbf{c}}) \xrightarrow{\sim} Z(\mathbf{e}H_{t,\mathbf{c}}\mathbf{e})$ for all parameters $(t, \mathbf{c})$.*

Proof. Clearly $z \mapsto z \cdot \mathbf{e}$ is a morphism $Z(H_{t,\mathbf{c}}) \to Z(\mathbf{e}H_{t,\mathbf{c}}\mathbf{e})$. Right multiplication on $H_{t,\mathbf{c}} \cdot \mathbf{e}$ by an element a in $Z(\mathbf{e}H_{t,\mathbf{c}}\mathbf{e})$ defines a right $\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$-linear endomorphism of $H_{t,\mathbf{c}} \cdot \mathbf{e}$. Therefore Theorem 2.4 says that there exists some $\zeta(a) \in H_{t,\mathbf{c}}$ such that right multiplication by a equals left multiplication on $H_{t,\mathbf{c}} \cdot \mathbf{e}$ by $\zeta(a)$. The action of a on the right commutes with left multiplication by any element of $H_{t,\mathbf{c}}$ hence $\zeta(a) \in Z(H_{t,\mathbf{c}})$. The homomorphism $\zeta \,:\, Z(\mathbf{e}H_{t,\mathbf{c}}\mathbf{e}) \to Z(H_{t,\mathbf{c}})$ is the inverse to the Satake isomorphism. $\square$

When $t = 0$, the Satake isomorphism becomes $Z_{0,\mathbf{c}} \xrightarrow{\sim} \mathbf{e}H_{t,\mathbf{c}}\mathbf{e}$ and is in fact an isomorphism of Poisson algebras.

Theorem 2.6. *The centre of the symplectic reflection algebra $H_{t,\mathbf{c}}(G)$ is described as follows:*

1. *If $t = 0$ then the Satake isomorphism identifies $Z_{\mathbf{c}}(G) := Z(H_{t,\mathbf{c}}(G)) \xrightarrow{\sim} \mathbf{e}H_{0,\mathbf{c}}\mathbf{e}$ and hence $H_{0,\mathbf{c}}(G)$ is a finite module over $Z_{\mathbf{c}}(G)$.*

2. *If $t \neq 0$ then $Z(H_{t,\mathbf{c}}(G)) = \mathbb{C}$.*

Proof. We have already proved statement (1). Statement (2) is due to Brown and Gordon [17, Proposition 7.2]. $\square$

Let (V, ω) is a symplectic vector space, $G \subset Sp(V)$ a finite group, and Γ the subgroup generated by all the symplectic reflections in G. It is a normal subgroup of G. As noted in [14, §4.2], the defining relations of $H_{t,\mathbf{c}}(G)$ show that $H_{t,\mathbf{c}}(G) \simeq H_{t,\mathbf{c}}(\Gamma) \rtimes (G/\Gamma)$. Therefore the only interesting algebra deformations occur when G is a symplectic reflection group.

2.3 The rational Cherednik algebra

There is a standard way to construct a large number of symplectic reflection groups - by creating them out of complex reflection groups. So let W be a complex reflection group, acting on the vector space $\mathfrak{h}$. Then W acts diagonally on $\mathfrak{h} \times \mathfrak{h}^*$. The space $\mathfrak{h} \times \mathfrak{h}^*$ has a natural pairing $(\cdot, \cdot) : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ defined by $(y, x) = x(y)$, and

$$\omega((x_1, y_1), (x_2, y_2)) := (y_2, x_1) - (y_1, x_2)$$

defines a W-equivariant symplectic form on $\mathfrak{h} \times \mathfrak{h}^*$. Therefore W acts on the symplectic space $\mathfrak{h} \times \mathfrak{h}^*$ as a symplectic reflection group and one can easily check that $(\mathfrak{h} \times \mathfrak{h}^*, \omega, W)$ is an indecomposable triple if and only if $\mathfrak{h}$ is a simple W-module, that is, W is an irreducible complex reflection group. Denote by $\mathcal{S}(W)$ the set of all complex reflections in W; it is also the set of symplectic reflections in W when considered as a symplectic reflection group. The *rational Cherednik algebra* $H_{t,\mathbf{c}}(W)$, as introduced by Etingof and Ginzburg [36, page 250], is the symplectic reflection algebra associated to the indecomposable triple $(\mathfrak{h} \times \mathfrak{h}^*, \omega, W)$. In this particular situation it is convenient to try and simplify the defining relations (2.1) a little. For $s \in \mathcal{S}(W)$, fix $\alpha_s \in \mathfrak{h}^*$ to be a basis of the one dimensional space $\mathrm{Im}\,(s-1)|_{\mathfrak{h}^*}$ and $\alpha_s^\vee \in \mathfrak{h}$ a basis of the one dimensional space $\mathrm{Im}\,(s-1)|_{\mathfrak{h}}$, normalized so that $\alpha_s(\alpha_s^\vee) = 2$. Then the relations (2.1) can be expressed as:

$$[x_1, x_2] = 0, \qquad [y_1, y_2] = 0, \qquad [x_1, y_1] = t(y_1, x_1) - \sum_{s \in \mathcal{S}} \mathbf{c}(s)(y_1, \alpha_s)(\alpha_s^\vee, x_1)s, \tag{2.2}$$

for all $x_1, x_2 \in \mathfrak{h}^*$ and $y_1, y_2 \in \mathfrak{h}$. For any $\nu \in \mathbb{C}\backslash\{0\}$, the algebras $H_{\nu t, \nu \mathbf{c}}(W)$ and $H_{t,\mathbf{c}}(W)$ are isomorphic. In the thesis we will only consider the case $t = 0$, therefore we are free to rescale $\mathbf{c}$ by ν whenever this is convenient. Unlike a general symplectic reflection algebra, one can see from the relations (2.2) that setting $\deg(\mathfrak{h}^*) = 1, \deg(\mathfrak{h}) = -1$ and $\deg(W) = 0$ makes the rational Cherednik algebra $H_{t,\mathbf{c}}(W)$ into a $\mathbb{Z}$-graded algebra. This grading will be important later.

2.4 The generalized Calogero-Moser Space

The (classical) Calogero-Moser space was introduced by Kazhdan, Kostant and Sternberg [66] and studied further by Wilson in the wonderful paper [102]. Calogero [20] studied the integrable system describing the motion of n massless particles on the real line with a repulsive force between each pair of particles, proportional to the square of the distance between them. In [66], Kazhdan, Kostant and Sternberg give a description of the corresponding phase space in terms of Hamiltonian reduction. By considering the real line as being the imaginary axis sitting in the complex plane, Wilson interprets the Calogero-Moser phase space as an affine variety

$$\mathcal{C}_n = \{(X, Y; u, v) \in \mathrm{Mat}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C}) \times \mathbb{C}^n \times (\mathbb{C}^n)^* \mid [X, Y] + I_n = v \cdot u\}//GL_n(\mathbb{C}). \tag{2.3}$$

He showed, [102, §1], that $\mathcal{C}_n$ is a smooth, irreducible, symplectic affine variety. For further reading see [35]. The relation to rational Cherednik algebras comes from an isomorphism by Etingof and Ginzburg between the affine variety $X_1(S_n) = \mathrm{Spec}\,(Z(H_{0,1}(S_n)))$, here S_n denotes the symmetric group on n objects, and the Calogero-Moser space $\mathcal{C}_n$:

$$\psi_n : X_1(S_n) \xrightarrow{\sim} \mathcal{C}_n.$$

It is an isomorphism of affine symplectic varieties and implies that $X_{\mathbf{c}}(S_n)$ is smooth when $\mathbf{c} \neq 0$.

The centre $Z_{\mathbf{c}}(W)$ of $H_{0,\mathbf{c}}(W)$ is an affine domain. We shall denote by $X_{\mathbf{c}}(W) = \mathrm{Spec}\,(Z_{\mathbf{c}}(W))$, the corresponding affine variety. Based upon the isomorphisms ψ_n above, the space $X_{\mathbf{c}}(W)$ is called the *generalized Calogero-Moser space* associated to the complex reflection group W at parameter $\mathbf{c}$. For $t \neq 0$, Etingof and Ginzburg [36, §4] showed that Dunkl operators define an embedding $H_{1,\mathbf{c}}(W) \hookrightarrow \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}) \rtimes W$, where $\mathfrak{h}_{\mathrm{reg}}$ is open subset of $\mathfrak{h}$ on which W acts freely. Carefully taking the limit $t \to 0$ shows that $H_{0,\mathbf{c}}(W) \hookrightarrow \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}] \rtimes W$. It is clear from this embedding that $\mathbb{C}[\mathfrak{h}]^W \subset Z_{\mathbf{c}}(W)$. Using a "Fourier automorphism" of $H_{0,\mathbf{c}}(W)$, swapping $\mathfrak{h}$ and $\mathfrak{h}^*$, one can show that $\mathbb{C}[\mathfrak{h}^*]^W \subset Z_{\mathbf{c}}(W)$ too. Therefore we arrive at:

Proposition 2.7 (Proposition 4.15, [36])**.** *Let $H_{0,\mathbf{c}}(W)$ be a rational Cherednik algebra associated to the complex reflection group W.*

1. *The subalgebra $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ of $H_{0,\mathbf{c}}(W)$ is contained in $Z_{\mathbf{c}}(W)$.*
2. *The centre $Z_{\mathbf{c}}(W)$ of $H_{0,\mathbf{c}}(W)$ is a free $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$-module of rank $|W|$.*

A more direct proof of the above Proposition is given in [48, Proposition 3.6]. The inclusions $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow Z_{\mathbf{c}}(W)$ and $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_{\mathbf{c}}(W)$ define surjective morphisms

$$\pi_1 : X_{\mathbf{c}}(W) \twoheadrightarrow \mathfrak{h}^*/W \qquad \text{and} \qquad \pi_2 : X_{\mathbf{c}}(W) \twoheadrightarrow \mathfrak{h}/W.$$

We write

$$\Upsilon : X_{\mathbf{c}}(W, \mathfrak{h}) \twoheadrightarrow \mathfrak{h}^*/W \times \mathfrak{h}/W$$

for the product morphism $\Upsilon = \pi_1 \times \pi_2$. It is a finite and hence closed, surjective morphism. Since $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ is an affine algebra, the Artin-Tate Lemma [74, Lemma 13.9.10] gives another proof of the fact that $Z_{\mathbf{c}}(W)$ is affine.

We note here the compatibility of the maps π_1 and π_2 in the case $W = S_n$ with the isomorphism ψ_n above. Wilson [102, Corollary 1.5] showed that the action of $GL_n(\mathbb{C})$ in (2.3) is free. Therefore a point $p \in \mathcal{C}_n$ corresponds to an orbit $GL_n(\mathbb{C}) \cdot (X, Y)$. We can define a map, which we will denote by the same symbol Υ, $\mathcal{C}_n \to \mathbb{C}^n/S_n \times (\mathbb{C}^n)^*/S_n$ that takes the pair (X, Y) to $(\mathrm{Eigenvalues}\,(X), \mathrm{Eigenvalues}\,(Y))$.

Then one can check that the following diagram is commutative.

$$\begin{tikzcd} X_1(S_n) \arrow[rr, "\psi_n"] \arrow[d, "\Upsilon"'] & & \mathcal{C}_n \arrow[d, "\Upsilon"] \\ \mathbb{C}^n/S_n \times (\mathbb{C}^n)^*/S_n \arrow[rr, "\mathrm{id}"] & & \mathbb{C}^n/S_n \times (\mathbb{C}^n)^*/S_n \end{tikzcd}$$

This observation will be useful in chapter 5.

As we have mentioned several times already, Poisson structures will play an important role in the study of the representation theory of rational Cherednik algebras at $t = 0$. Therefore it will come as no surprise to the reader that the centre $Z_{\mathbf{c}}(W)$ of $H_{0,\mathbf{c}}(W)$ is a Poisson algebra. Let us describe the bracket $\{-,-\}$ on $Z_{\mathbf{c}}(W)$. Consider the rational Cherednik algebra $H_{\mathbf{t},\mathbf{c}}(W)$, where $\mathbf{t}$ is a central indeterminate. It is a $\mathbb{C}[\mathbf{t}]$-algebra and there is a canonical isomorphism

$$\rho\,:\, H_{\mathbf{t},\mathbf{c}}(W)/\mathbf{t}\cdot H_{\mathbf{t},\mathbf{c}}(W) \xrightarrow{\sim} H_{0,\mathbf{c}}(W).$$

Since the centre $Z_{\mathbf{c}}(W)$ of $H_{0,\mathbf{c}}(W)$ is an affine domain over which $H_{0,\mathbf{c}}(W)$ is a finite module we are in the situation described in (1.3). Hence $Z_{\mathbf{c}}(W)$ is a Poisson algebra. This Poisson structure is particular nice, as illustrated by:

Theorem 2.8 (Theorem 7.8, [17])**.** *The symplectic leaves of the Poisson variety $X_{\mathbf{c}}(W)$ are precisely the smooth points of the irreducible components of the rank stratification. In particular they are finite in number, hence the bracket $\{-,-\}$ is algebraic.*

2.5 The restricted rational Cherednik algebra

The inclusion of algebras $A := \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_{\mathbf{c}}(W)$ allows us to define the *restricted rational Cherednik algebra* $\bar{H}_{\mathbf{c}}(W)$ as

$$\bar{H}_{\mathbf{c}}(W) = \frac{H_{\mathbf{c}}(W)}{A_+ \cdot H_{\mathbf{c}}(W)},$$

where A_+ denotes the ideal in A of elements with zero constant term. This algebra was originally introduced and extensively studied in the paper [48]. The PBW theorem implies that

$$\bar{H}_{\mathbf{c}}(W) \cong \mathbb{C}[\mathfrak{h}]^{coW} \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*]^{coW}$$

as vector spaces. Since W is a complex reflection group, Proposition 1.13 implies that $\dim \bar{H}_{\mathbf{c}}(W) = |W|^3$. The representation theory of $\bar{H}_{\mathbf{c}}(W)$ will play an important role throughout this thesis. We denote by $\mathsf{Irr}(W)$ a set of complete, non-isomorphic simple W-modules. Following [48]:

Definition 2.9. Let $\lambda \in \mathsf{Irr}(W)$. The *baby Verma module* of $\bar{H}_{\mathbf{c}}(W)$ associated to λ is

$$\Delta(\lambda) := \bar{H}_{\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes W} \lambda,$$

where $\mathbb{C}[\mathfrak{h}^*]^{coW}_+$ acts on λ as zero.

If M is a right $\bar{H}_{\mathbf{c}}(W)$ then M^* becomes a left $\bar{H}_{\mathbf{c}}(W)$-module, where the action of $\bar{H}_{\mathbf{c}}(W)$ on M^* is defined to be

$$(h \cdot f)(m) := f(m \cdot h) \quad \forall\, h \in \bar{H}_{\mathbf{c}}(W), m \in M, f \in M^*.$$

Definition 2.10. Let $\lambda \in \mathsf{Irr}(W)$. The *dual baby Verma module* of $\bar{H}_{\mathbf{c}}(W)$ associated to λ is

$$\nabla(\lambda) := \left(\lambda^* \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \rtimes W} \bar{H}_{\mathbf{c}}(W)\right)^*,$$

where $\mathbb{C}[\mathfrak{h}]^{coW}_+$ acts on λ^* as zero.

The ideal $A_+ \cdot H_{\mathbf{c}}(W)$ is a graded ideal in $H_{\mathbf{c}}(W)$. Therefore the restricted rational algebra is also a $\mathbb{Z}$-graded algebra. The baby Verma modules and the dual baby Verma modules can also be considered as $\mathbb{Z}$-graded modules. Let us denote by $\bar{H}_{\mathbf{c}}(W)$-mod$_{\mathbb{Z}}$ the category of finitely generated, graded $\bar{H}_{\mathbf{c}}(W)$-modules. The morphisms in $\bar{H}_{\mathbf{c}}(W)$-mod$_{\mathbb{Z}}$ are graded morphisms of degree zero. If $M \in \bar{H}_{\mathbf{c}}(W)$-mod$_{\mathbb{Z}}$ then $M[i]$ will denote the graded $\bar{H}_{\mathbf{c}}(W)$-module with grading $M[i]_j = M_{j-i}$, where $i, j \in \mathbb{Z}$. We denote by F the forgetful functor $\bar{H}_{\mathbf{c}}(W)$-mod$_{\mathbb{Z}} \to \bar{H}_{\mathbf{c}}(W)$-mod. We summarize, as is done in [48, Proposition 4.3] the results of [56] applied to this situation.

Proposition 2.11 (Proposition 4.3, [48])**.** *Let $\lambda, \mu \in$ Irr(W).*

- *The baby Verma module $\Delta(\lambda)$ has a simple head, $L(\lambda)$. Hence $\Delta(\lambda)$ is indecomposable.*
- *$\Delta(\lambda)$ is isomorphic to $\Delta(\mu)[i]$ if and only if $i = 0$ and $\lambda \simeq \mu$.*
- *The set $\{\, L(\lambda)[i] \mid \lambda \in$ Irr$(W), i \in \mathbb{Z}\}$ is a complete set of pairwise non-isomorphic simple graded $\bar{H}_{\mathbf{c}}(W)$-modules.*
- *$F(L(\lambda))$ is a simple $\bar{H}_{\mathbf{c}}(W)$-module and $\{\, F(L(\lambda)) \mid \lambda \in$ Irr$(W)\}$ is a complete set of pairwise non-isomorphic simple $\bar{H}_{\mathbf{c}}(W)$-modules.*
- *If $P(\lambda)$ is the projective cover of $L(\lambda)$ in $\bar{H}_{\mathbf{c}}(W)$-mod$_{\mathbb{Z}}$ then $F(P(\lambda))$ is the projective cover of $F(L(\lambda))$ in $\bar{H}_{\mathbf{c}}(W)$-mod.*

2.6 The Calogero-Moser partition

Following [51] we define the *Calogero-Moser partition* of Irr $\bar{H}_{\mathbf{c}}(W)$ to be the set of equivalence classes of Irr $\bar{H}_{\mathbf{c}}(W)$ under the equivalence relation $L \sim M$ if and only if L and M belong to the same block of $\bar{H}_{\mathbf{c}}(W)$. The set of equivalence classes will be denoted $\mathsf{CM}_{\mathbf{c}}(W)$. Since the map $\lambda \mapsto L(\lambda)$ naturally identifies $\mathsf{Irr}(W)$ with Irr $\bar{H}_{\mathbf{c}}(W)$, the Calogero-Moser partition $\mathsf{CM}_{\mathbf{c}}(W)$ can (and shall) be thought of as a partition of $\mathsf{Irr}(W)$. Given $\lambda, \mu \in \mathsf{Irr}(W)$ we say that λ, μ belong to the same partition of $\mathsf{CM}_{\mathbf{c}}(W)$ if they are in the same equivalence class.

It is a consequence of a theorem by Müller, [75, Theorem 7] (see [18, Corollary 2.7] for a formulation relevant to our situation) that the primitive central idempotents of $\bar{H}_{\mathbf{c}}(W)$ are the images of the primitive idempotents of $Z_{\mathbf{c}}/A_+ \cdot Z_{\mathbf{c}}$ under the natural map $Z_{\mathbf{c}}/A_+ \cdot Z_{\mathbf{c}} \to \bar{H}_{\mathbf{c}}(W)$. This means that the natural map $\mathsf{Irr}(W) \to \Upsilon^{-1}(0)$, $\lambda \mapsto \mathrm{Supp}\,(L(\lambda))$, factors through the Calogero-Moser partition (here $\Upsilon^{-1}(0)$ is considered as the set theoretic pull-back):

$$\begin{tikzcd}
\mathsf{Irr}(W) \arrow[d, two heads] \arrow[dr] & \\
\mathsf{CM}_{\mathbf{c}}(W) \arrow[r, dotted, "\sim"] & \Upsilon^{-1}(0)
\end{tikzcd}$$

2.7 Example: The cyclic group

In this case we fix a basis $\mathfrak{h}^* = \mathbb{C} \cdot x$ and $\mathfrak{h} = \mathbb{C} \cdot y$ such that the action of the cyclic group $C_m = \langle \varepsilon \rangle$ is given by $\varepsilon \cdot x = \zeta x$ and $\varepsilon \cdot y = \zeta^{-1} y$, where ζ is a fix primitive m^{th} root of unity. If we fix $\alpha_{\varepsilon^i} = \sqrt{2} \cdot x$ and $\alpha_{\varepsilon^i}^\vee = \frac{-1}{\sqrt{2}} \cdot y$ then the commutation relations defining $H_{t,\mathbf{c}}(C_m)$ are:

$$\begin{aligned}
\varepsilon \cdot x &= \zeta x \cdot \varepsilon \\
\varepsilon \cdot y &= \zeta^{-1} y \cdot \varepsilon \\
[y, x] &= t + \textstyle\sum_{i=1}^{m-1} c_i \varepsilon^i.
\end{aligned}$$

The idempotents in $\mathbb{C}\, C_m$ corresponding to the simple C_m-modules are $e_i = \frac{1}{m} \sum_{j=0}^{m-1} \zeta^{-ij} \varepsilon^j$, $0 \le i \le m-1$ so that $\varepsilon \cdot e_i = \zeta^i e_i$. Then $e_{i+1} \cdot x = x \cdot e_i$ and $e_{i-1} \cdot y = y \cdot e_i$. The centre of $H_{0,\mathbf{c}}(C_m)$ is generated by the three elements

$$\bar{A} := x^m, \quad \bar{B} := xy + \sum_{i=1}^{m-1} \frac{c_i}{1 - \zeta^i} \varepsilon^i, \quad \bar{C} := \quad y^m.$$

The centre is isomorphic to the affine variety

$$\mathbb{C}[A, B, C]/(AC - f(B)),$$

where

$$f(B) = \prod_{i=0}^{m-1} (B - \sum_{j=1}^{m-1} \frac{c_j \cdot \zeta^{ij}}{1 - \zeta^j}),$$

and the isomorphism is given by $\bar{A} \mapsto A$, $\bar{B} \mapsto B$ and $\bar{C} \mapsto C$. There is one two-dimensional leaf in $X_{\mathbf{c}}$, corresponding to the smooth locus of $X_{\mathbf{c}}$. The singular locus will consist of a finite number of isolated points. These points are the zero dimensional leaves of $X_{\mathbf{c}}$. They are in bijection with the roots of f that have multiplicity greater than one. Therefore $X_{\mathbf{c}}$ is smooth if and only if the roots of f are all distinct. The function $\mathbf{c}$ can be considered a class function on C_m (by extending by zero). One can rewrite $\mathbf{c}$ as a function on the simple C_m-modules and hence write the defining relations of $H_{t,\mathbf{c}}(C_m)$ in terms of idempotents in the group algebra. Let $(H_0, \ldots, H_{m-1}) \in \mathbb{C}^m$ such that $\sum_{i=0}^{m-1} H_i = 0$ and relate them to $\mathbf{c}$ by $c_i = \sum_{j=0}^{m-1} \zeta^{-ij} H_j$. The main commutation relation for $H_{t,\mathbf{c}}(C_m)$ becomes

$$[y, x] = t + m \sum_{i=0}^{m-1} H_i e_i.$$

Then $B = xy + m\sum_{i=1}^{m-1}(\sum_{j=0}^{i-1} H_j)e_i$. As explained in (2.6), the Calogero-Moser partition of $\bar{H}_{\mathbf{c}}(C_m)$ can be considered a partition of the set $\mathsf{Irr}\,(C_m) = \{e_0, \ldots, e_{m-1}\}$. In this case, e_i and e_j will be in the same block of the Calogero-Moser partition if and only if the central element B acts on $L(e_i)$ and $L(e_j)$ as the same scalar. Using the fact that B acts on the baby Verma module $\Delta(e_i)$ as the same scalar as on $L(e_i)$ one can calculate that e_i and e_j are in the same block of the Calogero-Moser partition if and only if $\sum_{k=0}^{i-1} H_k = \sum_{k=0}^{j-1} H_k$. In fact the identity

$$[y, x^j] = x^{j-1}\sum_{i=1}^{m-1}\left(\sum_{k=0}^{j-1} H_{i+k}\right)e_i \quad \forall\, j \geq 1$$

implies that

$$(\Delta(e_i) : L(e_{i+j})[l]) = \begin{cases} 1 & \text{if } l = j \text{ and } \sum_{k=0}^{j-1} H_{i+k} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2.8 Parabolic subgroups

Finally, we would just like to mention a few technical facts about parabolic subgroups of complex reflection groups. These facts will be important in chapter 5. We fix a complex reflection group W and let W' be a subgroup of W. It is called a *parabolic subgroup* if there is a set $S \subseteq \mathfrak{h}$ such that $W' = \mathrm{Stab}_W(S)$. Since W acts linearly on $\mathfrak{h}$ every parabolic subgroup is the stabilizer of some linear subspace of $\mathfrak{h}$. By a theorem of Steinberg [94, Theorem 1.5], a parabolic subgroup is itself a complex reflection group. Note that, in general, there exist subgroups of W that are themselves complex reflection groups but are not parabolic subgroups e.g. $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}$. We write

$$(\mathfrak{h}^{*W'})^{\perp} := \{y \in \mathfrak{h} \,|\, x(y) = 0 \text{ for all } x \in \mathfrak{h}^{*W'}\}.$$

Then $\mathfrak{h} = \mathfrak{h}^{W'} \oplus (\mathfrak{h}^{*W'})^{\perp}$ is a decomposition of $\mathfrak{h}$ as a W'-module. Define the rank of a complex reflection group W' to be the dimension of a faithful reflection representation of W' of minimal dimension. Note that $(\mathfrak{h}^{*W'})^{\perp}$ is a faithful reflection representation of W' of minimal dimension, hence the rank of W' is $\dim(\mathfrak{h}^{*W'})^{\perp}$. When W is a real reflection group this definition of rank agrees, by [59, Theorem 1.12], with the alternative definition of rank in terms of root systems ([59, 1.3]). The group W acts on its set of parabolic subgroups by conjugation. Given a parabolic subgroup W', the corresponding conjugacy class will be denoted (W'). We also require the partial ordering on conjugacy classes of parabolic subgroups of W defined by $(W_1) \geq (W_2)$ if and only if W_1 is conjugate to a subgroup of W_2 (the ordering is chosen in this way so that it agrees with a geometric ordering to be introduced in Section 5.2). Finally, for a given parabolic subgroup W' of W we denote by $\mathfrak{h}_{\mathrm{reg}}^{W'}$ the subset of $\mathfrak{h}^{W'}$ consisting of those points whose stabilizer is W': it is a locally closed subset of $\mathfrak{h}$.

2.9 Remarks

1. Drinfeld had already proved the PBW theorem, Theorem 2.1, in the paper [32] but this result was unknown to Etingof and Ginzburg until recently.

2. The best reference for symplectic reflection algebras is still the paper [36] in which they were first introduced. It contains many original ideas. For a survey of developments since then see [50].

3. The results and definitions regarding restricted rational Cherednik algebras were taken from [48].

Chapter 3

Singular Calogero-Moser spaces

In this chapter we explore one aspect of the interplay between the geometry of the generalized Calogero-Moser space and the representation theory of the rational Cherednik algebra. By studying the representation theory of the restricted rational Cherednik algebra, we show that it is possible to classify the irreducible complex reflection groups whose associated generalized Calogero-Moser space is singular for all values of the deformation parameter $\mathbf{c}$. Using combinatorial properties of complex reflection groups we show that if the group W is different from the wreath product $C_m \wr S_n$ and the binary tetrahedral group (labeled $G(m,1,n)$ and G_4 respectively in the Shephard-Todd classification), then the generalized Calogero-Moser space $X_{\mathbf{c}}$ associated to the centre of the rational Cherednik algebra $H_{0,\mathbf{c}}(W)$ is singular for all $\mathbf{c}$. In the next chapter we will see how this enables us to deduce the existence or otherwise of symplectic resolutions of the singular symplectic variety $\mathfrak{h} \times \mathfrak{h}^*/W$. Therefore the goal of this chapter is to prove the following theorem:

Theorem 3.1. *Let W be an irreducible complex reflection group, not isomorphic to $G(m,1,n)$ or G_4, and $X_{\mathbf{c}}$ the generalized Calogero-Moser space associated to W. Then $X_{\mathbf{c}}$ is a singular variety for all choices of the parameter $\mathbf{c}$. Conversely for $W \simeq G_4$, $X_{\mathbf{c}}$ is a smooth variety for generic values of $\mathbf{c}$.*

By describing the generalized Calogero-Moser space associated to the wreath product $C_m \wr S_n$ as an affine quiver variety, Etingof and Ginzburg [36, Corollary 1.14] have shown that, for generic values of the parameter $\mathbf{c}$, $X_{\mathbf{c}}$ is smooth. On the other hand, Gordon [48, Proposition 7.3] showed that, for almost all Weyl groups W not of type A or B $(= C)$, $X_{\mathbf{c}}$ is a singular variety for all choices of the parameter $\mathbf{c}$. We use arguments similar to those used by Gordon in order to prove Theorem 3.1.

3.1 Singular points on $X_{\mathbf{c}}$

The geometry of the generalized Calogero-Moser space is encoded in the representation theory of the corresponding rational Cherednik algebra. In particular, a closed point of $X_{\mathbf{c}}$ is singular if and only if there is a "small" simple module supported at that point (this statement is made precise in Proposition 3.2 below). Therefore we will prove Theorem 3.1 by hunting for these "small" simple modules. Since the algebra $H_{\mathbf{c}}$ is a finite module over $Z_{\mathbf{c}}$ it is a P.I. (polynomial identity) ring ([74, Corollary 13.1.13]).

It is a consequence of Kaplansky's Theorem that every finitely generated simple $H_{\mathbf{c}}$-module is a finite dimensional vector space over $\mathbb{C}$. More precisely, if L is a simple $H_{0,\mathbf{c}}(W)$-module then $m = \dim L \leq$ $P.I.\,degree\,(H_{0,\mathbf{c}}(W))$ and $H_{0,\mathbf{c}}(W)/\mathrm{ann}_{H_{0,\mathbf{c}}(W)}\, L \simeq \mathrm{Mat}_m(\mathbb{C})$. Schur's lemma says that the elements of the centre $Z_{\mathbf{c}}$ of $H_{\mathbf{c}}$ act as scalars on any simple $H_{\mathbf{c}}$-module L. Therefore the simple module L defines a character $\chi_L : Z_{\mathbf{c}} \to \mathbb{C}$ and the kernel of χ_L is a maximal ideal in $Z_{\mathbf{c}}$, or equivalently a closed point in $X_{\mathbf{c}}$. Without loss of generality we will refer to this point as χ_L and denote by $Z_{\mathbf{c}}(W)_{\chi_L}$ the localization of $Z_{\mathbf{c}}(W)$ at the maximal ideal Ker χ_L. We denote by $H_{0,\mathbf{c}}(W)_\chi$ the central localization $H_{0,\mathbf{c}}(W) \otimes_{Z_{\mathbf{c}}(W)} Z_{\mathbf{c}}(W)_\chi$. The Azumaya locus of $H_{0,\mathbf{c}}(W)$ over $Z_{\mathbf{c}}(W)$ is defined to be

$$\mathcal{A}_{\mathbf{c}} := \{\chi \in X_{\mathbf{c}}(W) \,|\, H_{0,\mathbf{c}}(W)_\chi \text{ is Azumaya over } Z_{\mathbf{c}}(W)_\chi\}.$$

As shown in [16, Theorem III.1.7], $\mathcal{A}_{\mathbf{c}}$ is a non-empty, open subset of $X_{\mathbf{c}}(W)$.

Proposition 3.2. *Let L be a simple $H_{\mathbf{c}}$-module then* $\dim L = |W|$ *if and only if χ_L is a nonsingular point of $X_{\mathbf{c}}$.*

Proof. It is a consequence of the Artin-Procesi Theorem [74, Theorem 13.7.14] that the following are equivalent:

1. $\chi \in \mathcal{A}_{\mathbf{c}}$;

2. $\dim L = P.I.\,degree\,(H_{0,\mathbf{c}}(W))$ for all simple modules L such that $\chi_L = \chi$;

3. there exists a unique simple module L such that $\chi_L = \chi$.

Lemma 2.2 shows that the dimension of a generic simple module is $|W|$. Since the Azumaya locus $\mathcal{A}_{\mathbf{c}}$ is dense in $X_{\mathbf{c}}$ it follows that $P.I.degree\,(H_{0,\mathbf{c}}(W)) = |W|$. The proposition will then follow from the statement $\mathcal{A}_{\mathbf{c}} = (X_{\mathbf{c}})_{sm}$, where $(X_{\mathbf{c}})_{sm}$ is the smooth locus of $X_{\mathbf{c}}$. Since the skew group ring $\mathbb{C}[\mathfrak{h}\times\mathfrak{h}^*]\rtimes W$, which by Proposition 1.14 has global dimension $2\cdot\dim\,\mathfrak{h}$, is the associated graded of $H_{0,\mathbf{c}}(W)$, the result [74, Corollary 7.6.18] says that gl.dim$\,(H_{0,\mathbf{c}}(W)) \leq 2\cdot\dim\,\mathfrak{h}$. Then the result [16, Lemma III.1.8] says that gl.dim $(H_{0,\mathbf{c}}(W)) < \infty$ implies that $\mathcal{A}_{\mathbf{c}} \subseteq (X_{\mathbf{c}})_{sm}$. The opposite inclusion is an application of a result by Brown and Goodearl, [15, Theorem 3.8]. Their theorem says that $(X_{\mathbf{c}})_{sm} \subseteq \mathcal{A}_{\mathbf{c}}$ (in fact that we have equality) if $H_{0,\mathbf{c}}(W)$ has particularly nice homological properties - it must be Auslander-regular and Cohen-Macaulay, and the complement of $\mathcal{A}_{\mathbf{c}}$ has codimension at least two in $X_{\mathbf{c}}$. The fact that $H_{0,\mathbf{c}}(W)$ is Auslander-regular and Cohen-Macaulay can be deduced from the fact that it's associated graded, the skew group ring, has these properties (the results that are required to show this are listed in the proof of [14, Theorem 4.4]). The fact that the complement of $\mathcal{A}_{\mathbf{c}}$ has co-dimension at least two in $X_{\mathbf{c}}$ is harder to show. It follows from the fact that $X_{\mathbf{c}}$ is a symplectic variety, Proposition 4.13, and that the "representation theory of $H_{0,\mathbf{c}}$ is constant along orbits", (5.17). $\square$

An elegant, direct proof of the fact that χ_L a non-singular point of $X_{\mathbf{c}}$ implies that $\dim L = |W|$ was given by Etingof and Ginzburg [36, Theorem 1.7]. The proof depends on the study of the Etingof-Ginzburg sheaf on $X_{\mathbf{c}}$ which was introduced in (2.2). They show that the Etingof-Ginzburg sheaf is locally free at χ_L if and only if χ_L is smooth. Then Proposition 3.2 follows from the fact that L can be identified with the fiber of the Etingof-Ginzburg sheaf at χ_L when χ_L is smooth. We will study this sheaf further in chapter 5.

3.2 Proof of the main theorem

By Proposition 3.2, the first statement in Theorem 3.1 is equivalent to the result:

Proposition 3.3. *For each W not isomorphic to $G(m,1,n)$ or G_4, there exists an irreducible W-module λ such that for all parameters $\mathbf{c}$, the irreducible $\bar{H}_{0,\mathbf{c}}(W)$-module $L(\lambda)$ has dimension $< |W|$.*

The proof of Proposition 3.3 will occupy the whole of section 3.4. For an arbitrary finite dimensional $\mathbb{Z}$-graded vector space $M = \oplus_{i\in\mathbb{Z}} M_i$, we denote the Poincaré polynomial of M by $P(M,t)$. Denote by $f_\lambda(t)$ the *fake polynomial* of the $\lambda \in Irr(W)$. This is defined as

$$f_\lambda(t) := \sum_{i\in\mathbb{Z}_{\geq 0}} (\mathbb{C}[\mathfrak{h}]_i^{coW} : \lambda) t^i,$$

where $(\mathbb{C}[\mathfrak{h}]_i^{coW} : \lambda)$ is the multiplicity of λ in i^{th} degree of the co-invariant ring $\mathbb{C}[\mathfrak{h}]^{coW}$ (thought of here as a graded W-module).

Lemma 3.4. *Let W be a complex reflection group. Assume that $\lambda \in Irr(W)$ is chosen so that the support of $L(\lambda)$ is a smooth point in $\Upsilon^{-1}(0) \subset X_{\mathbf{c}}$. Then the Poincaré polynomial of $L(\lambda)$ as a graded vector space is given by*

$$P(L(\lambda),t) = \frac{\dim(\lambda) t^{b_{\lambda^*}} P(\mathbb{C}[\mathfrak{h}^*]^{coW}, t)}{f_{\lambda^*}(t)}, \tag{3.1}$$

where λ^* is the irreducible W-module dual to λ, and b_λ the valuation of the fake polynomial $f_\lambda(t)$.

Proof. By [48, Lemma 4.4, paragraphs (5.2) and (5.4)], the graded composition factors of $M(\lambda)$ are all of the form $L(\lambda)[i]$, for some $i \geq 0$. Therefore we can find a multi-set $\{i_1, \dots i_k\}$ such that as a graded W-module

$$M(\lambda) \cong L(\lambda)[i_1] \oplus L(\lambda)[i_2] \oplus \cdots \oplus L(\lambda)[i_k].$$

Since the support of $L(\lambda)$ is a smooth point in $X_{\mathbf{c}}$, Proposition 3.2 says that $L(\lambda) \simeq \mathbb{C}W$ as a W-module. Hence it contains a unique copy of the trivial representation T. Assume this copy occurs in degree a in $L(\lambda)$. Then it will occur in degree $a - i_j$ in $L(\lambda)[i_j]$. As a graded W-module, $M(\lambda) \cong \mathbb{C}[\mathfrak{h}^*]^{coW} \otimes \lambda$. The fact that $[\tau \otimes \lambda : T] = \delta_{\tau\lambda^*}$ implies that the graded multiplicity of T in $M(\lambda)$ equals the graded multiplicity of λ^* in $\mathbb{C}[\mathfrak{h}^*]^{coW}$. The graded multiplicity of λ^* in $\mathbb{C}[\mathfrak{h}^*]^{coW}$ is $f_{\lambda^*}(t)$. Hence $P(M(\lambda),t) = t^{-a} f_{\lambda^*}(t) P(L(\lambda),t)$. The lowest nonzero term of $P(L(\lambda),t)$ occurs in degree zero implying that $a = b_{\lambda^*}$. The formula follows by noting that $P(M(\lambda),t)$ is $\dim(\lambda)P(\mathbb{C}[\mathfrak{h}^*]^{coW}, t)$. ☐

Since $L(\lambda)$ is a finite dimensional module, the above lemma shows that the right hand side of equation (3.1) is a polynomial in $\mathbb{N}[t,t^{-1}]$ with integer coefficients. Moreover, [48, Lemma 4.4] shows that it is actually in $\mathbb{N}[t]$ and that the degree 0 coefficient is $\dim\lambda$.

3.3 The infinite series $G(m, d, n)$

In this section we show that for $d \neq 1$ and $W = G(m, d, n) \neq G(2, 2, 3)$ it is always possible to choose an irreducible representation λ of $G(m, d, n)$ such that Lemma 3.4 does not hold. Thus $L(\lambda)$ will have dimension $< |G(m, d, n)|$, proving Proposition 3.3 in these cases. The group $G(2, 2, 3)$ is the Weyl group corresponding to the Dynkin diagram $D_3 = A_3$ and hence $G(2, 2, 3) \cong S_4$. By [36, Corollary 16.2], $X_{\mathbf{c}}$ is smooth for generic and hence all non-zero $\mathbf{c}$ in this case. Recall the definition of the imprimitive complex reflection groups $G(m, d, n)$ as given in (1.5). We fix $p = m/d$ and ζ a primitive m^{th} root of unity. Let $s_{(i,j)} \in S_n$ denote the transposition swapping i and j and let ε_i^k be the matrix in $A(m, 1, n)$ which has ones all along the diagonal except in the i^{th} position where it's entry is ζ^k.

We begin by giving an explicit description of the simple $G(m, 1, n)$-modules. A *partition* of n is a sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ such that $n = |\lambda| := \sum_{i=1}^{k} \lambda_k$. We call k the *length* of λ. The simple S_n-modules are parameterized by partitions of n. Let V_λ denote the simple S_n-module labeled by the partition λ. The simple C_m-modules will be denoted $\mathbb{C} \cdot \omega_i$ (or simply ω_i), $0 \leq i \leq m$. If $C_m = \langle \varepsilon \rangle$ then $\varepsilon \cdot \omega_i = \zeta^i \omega_i$ (we may think of $C_m \subset G(m, 1, n)$ such that $\varepsilon = \varepsilon_1$). Now let U be any C_m-module and V a S_n-module. The *wreath product* $U \wr V$ is the $G(m, 1, n)$-module, which as a vector space is $U^{\otimes n} \otimes V$, and whose module structure is uniquely defined by

$$\varepsilon_i \cdot (u_1 \otimes \cdots \otimes u_n \otimes v) = u_1 \otimes \cdots \otimes \varepsilon \cdot u_i \otimes \cdots \otimes u_n \otimes v, \tag{3.2}$$

and for $\sigma \in S_n$,

$$\sigma \cdot (u_1 \otimes \cdots \otimes u_n \otimes v) = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(n)} \otimes \sigma \cdot v.$$

If U and V are simple modules then $U \wr V$ is a simple $G(m, 1, n)$-module. However, not every simple $G(m, 1, n)$-module can be written in this way. A complete set of non-isomorphic simple modules was originally constructed by Specht [90], this result is given below and a proof can be found in [62, Theorem 4.3.34]. An m-multi-partition $\underline{\lambda}$ of n is an ordered m-tuple of partitions $(\lambda^0, \ldots, \lambda^{m-1})$ such that $|\lambda^0| + \cdots + |\lambda^{m-1}| = n$. Let $\mathcal{P}(m, n)$ denote the set of all m-multi-partitions of n. To each m-tuple $n_0 + \cdots + n_{m-1} = n$ we may associate the *Young subgroup* $G_{(n)} = C_m \wr (S_{n_0} \times \cdots \times S_{n_{m-1}})$ of $G(m, 1, n)$.

Theorem 3.5. *To each $\underline{\lambda}$ in $\mathcal{P}(m, n)$ we can associate the $G(m, 1, n)$-module*

$$V_{\underline{\lambda}} := \mathsf{Ind}_{G_{(n)}}^{G(m,1,n)} (\omega_0 \wr V_{\lambda^0}) \otimes \cdots \otimes (\omega_{m-1} \wr V_{\lambda^{m-1}}),$$

where $G_{(n)}$ is the Young subgroup associated to the m-tuple $|\lambda^0| + \cdots + |\lambda^{m-1}| = n$. Each $V_{\underline{\lambda}}$ is simple, $V_{\underline{\lambda}} \not\cong V_{\underline{\mu}}$ for $\underline{\lambda} \neq \underline{\mu}$ and every simple $G(m, 1, n)$-module is isomorphic to $V_{\underline{\lambda}}$ for some $\underline{\lambda}$.

Note that in the case $n_i = 0$, the module $\omega_i \wr V_i$ should be regarded as the one-dimensional trivial module. The group $G(m, d, n)$ is a normal subgroup of $G(m, 1, n)$ and the quotient group is the cyclic group C_d. We can realize $G(m, d, n)$ as the kernel of a linear character of $G(m, 1, n)$ as follows. An element of $G(m, 1, n)$ can be thought of as a permutation matrix but with the unique 1 in each row replaced by an element of C_m. The rule that takes each such matrix to the product of its non-zero entries defines a character $\delta' : G(m,1,n) \to \mathbb{C}^*$ (this is not the determinant of the matrix). Fix $\delta := (\delta')^p$. Then $G(m,d,n) = \mathrm{Ker}\ \delta$ and we can identify $C_d^\vee := \mathrm{Hom}_{gp}(C_d, \mathbb{C}^*) = \langle \delta \rangle$. It follows from (3.2) that $(\omega_i \wr V) \otimes \delta \simeq \omega_{i+p} \wr V$. If we define the action of $C_d^\vee$ on $\underline{\lambda}$ by

$$\delta \cdot (\lambda^0, \dots, \lambda^{m-1}) = (\lambda^{m-p}, \lambda^{m+1-p}, \dots, \lambda^{m-2}, \lambda^{m-1}, \lambda^0, \lambda^1, \dots, \lambda^{m-p-1}), \tag{3.3}$$

then Theorem 3.5 implies that $\delta \cdot V_{\underline{\lambda}} = V_{\delta \cdot \underline{\lambda}}$. We denote the orbit $C_d^\vee \cdot \underline{\lambda}$ by $\{\underline{\lambda}\}$. The stabilizer of $\underline{\lambda}$ in $C_d^\vee$ will be denoted $C_{\underline{\lambda}}^\vee$. We will see in the section on Clifford theory, (6.4), that there is an action of C_d on the set $\mathsf{Irr}\,(G(m,d,n))$ such that C_d acts transitively on the irreducible summands of $\mathsf{Res}\,^{G(m,1,n)}_{G(m,d,n)}\,\underline{\lambda}$. If μ is one of these summands then Proposition 6.11 says that μ has multiplicity one in $\mathsf{Res}\,^{G(m,1,n)}_{G(m,d,n)}\,\underline{\lambda}$ and $(C_d^\vee / C_{\underline{\lambda}}^\vee)^\vee = C_\mu \subset C_d$ is the stabilizer of μ with respect to the action of C_d. Therefore the irreducible summands of $\mathsf{Res}\,^{G(m,1,n)}_{G(m,d,n)}\,\underline{\lambda}$ are parametrized by elements of the quotient C_d/C_μ. This quotient can be identified with $C_{\underline{\lambda}}^\vee$ hence the set of all irreducible representations of $G(m,d,n)$ are parameterized by distinct pairs $(\{\underline{\lambda}\}, \epsilon)$, where $\epsilon \in C_{\underline{\lambda}}^\vee$. If we fix $C_d = \langle\, \overline{\varepsilon_1^p}\,\rangle$ and define the bijection $C_d \leftrightarrow C_d^\vee$ by $(\overline{\varepsilon_1^p})^i \leftrightarrow \delta^i$ then $C_d/C_\mu \leftrightarrow C_{\underline{\lambda}}^\vee$ and the action of C_d on pairs $(\{\underline{\lambda}\}, \epsilon)$ is given by

$$\eta \cdot (\{\underline{\lambda}\}, \epsilon) = (\{\underline{\lambda}\}, \eta \cdot \epsilon) \quad \text{where} \quad (\eta \cdot \epsilon)(\nu) = \epsilon(\eta\nu), \quad \text{for } \eta, \nu \in C_d. \tag{3.4}$$

3.4 Proof of Proposition 3.3

We will now try to calculate the Poincaré polynomial of $L(\lambda)$ for λ an irreducible $G(m,d,n)$-module. Let λ be a partition of n of length k and denote by $n(\lambda) = \sum_{i=1}^k (i-1)\lambda_i$ the partition statistic. The *Young diagram* of λ is defined to be the subset $Y(\lambda) := \{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le k,\ 1 \le j \le \lambda_i\}$ of $\mathbb{Z}^2$. Each box in the diagram is called a *node* and the *content* of a node (i,j) is defined to be the integer $\mathrm{cont}(i,j) := j - i$. The Young diagram should be visualized as a stack of boxes, justified to the left; for example the partition $(3,2,2,1)$ with its content is:

-2			
-1	0	1	
0	1	2	3

For $(i,j) \in Y(\lambda)$ we denote by $h(i,j)$ the hook length at (i,j), this is the number of boxes strictly above (i,j) plus the number of boxes strictly to the right of (i,j) plus one. For instance, the hook length of $(1,2)$ in the above Young diagram is 4. The hook polynomial is defined to be

$$H_\lambda(t) = \prod_{(i,j) \in Y(\lambda)} (1 - t^{h(i,j)}).$$

Let $(t)_{(n)} = (1-t)\cdots(1-t^{n-1})(1-t^n)$. The result [96, Corollary 6.4] says that the fake polynomial of the irreducible representation labeled by $(\{\underline{\lambda}\}, \epsilon)$ is

$$f_{\{\underline{\lambda}\}}(t) = \frac{1-t^{pn}}{1-t^{mn}} R_{\{\underline{\lambda}\}}(t) I_{\underline{\lambda}}(t^m), \tag{3.5}$$

where

$$R_{\{\underline{\lambda}\}}(t) = \sum_{\underline{\mu}\in\{\underline{\lambda}\}} t^{r(\underline{\mu})} \quad \text{with} \quad r(\underline{\mu}) = \sum_{i=0}^{m-1} i|\mu^i|, \qquad \text{and} \quad I_{\underline{\lambda}}(t) = (t)_{(n)} \prod_{i=0}^{m-1} \frac{t^{n(\lambda^i)}}{H_{\lambda^i}(t)}.$$

Note that the formula only depends on the orbit and not on the choice of stabilizer.

We wish to calculate the rational function (3.1) for a well chosen representation $(\{\underline{\mu}\},\epsilon)$ of $G(m,d,n)$. By [59, Theorem 3.15], the Poincaré polynomial of the co-invariant ring of W is given by

$$P(\mathbb{C}[\mathfrak{h}^*]^{coW}, t) = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t}$$

where $d_1,\dots,d_n$ are the degrees of a set of fundamental homogeneous invariant polynomials of W ($d_1,\dots,d_n$ are independent, up to reordering, of the choice of fundamental homogeneous invariants). By [88, page 291], $\{d_1,\dots,d_n\} = \{m, 2m, \dots, (n-1)m, pn\}$ when $W = G(m,d,n)$.

Lemma 3.6. *Let $(\{\underline{\lambda}\},\epsilon) \in \mathit{Irr}(G(m,d,n))$ be the unique representation corresponding to a smooth point of $\Upsilon^{-1}(0)$ in $X_{\mathbf{c}}$. Then the Poincaré polynomial of $L(\lambda)$ as a graded vector space is given by*

$$P(L(\{\underline{\mu}\},\epsilon),t) = \frac{\dim(\{\underline{\mu}\},\epsilon)\prod_{i=0}^{m-1} H_{\lambda^i}(t^m)}{(1-t)^n \tilde{R}_{\{\underline{\lambda}\}}(t)}. \tag{3.6}$$

where $(\{\underline{\lambda}\},\eta)$ labels the dual representation to $(\{\underline{\lambda}\},\epsilon)$ and $\tilde{R}$ is defined in the proof below.

Proof. In the setup of Lemma 3.6 equation (3.1) becomes

$$P(L(\{\underline{\mu}\},\epsilon),t) =$$

$$\frac{\dim(\{\underline{\mu}\},\epsilon)\, t^{b_{\{\underline{\lambda}\}}}\,(1-t^m)(1-t^{2m})\cdots(1-t^{(n-1)m})(1-t^{np})\prod_{i=0}^{m-1} H_{\lambda^i}(t^m)(1-t^{mn})}{(1-t)^n(1-t^{pn})\, R_{\{\underline{\lambda}\}}(t)(t^m)_{(n)}\prod_{i=0}^{m-1} t^{n(\lambda^i)m}}$$

$$= \frac{\dim(\{\underline{\mu}\},\epsilon)\, t^{b_{\{\underline{\lambda}\}}}\prod_{i=0}^{m-1} H_{\lambda^i}(t^m)}{(1-t)^n\, R_{\{\underline{\lambda}\}}(t)\prod_{i=0}^{m-1} t^{n(\lambda^i)m}}. \tag{3.7}$$

Let $k\in\mathbb{N}$ such that $t^k \mid R_{\{\underline{\lambda}\}}(t)$ but $t^{k+1} \nmid R_{\{\underline{\lambda}\}}(t)$ in $\mathbb{Z}[t]$ and write $R_{\{\underline{\lambda}\}}(t) = t^k\tilde{R}_{\{\underline{\lambda}\}}(t)$. Then rearrange equation (3.5) as

$$f_{\{\underline{\lambda}\}}(t) = \left(t^k\prod_{i=0}^{m-1} t^{n(\lambda^i)m}\right)\tilde{R}_{\{\underline{\lambda}\}}(t)\left(\frac{1-t^{pn}}{1-t^{mn}}(t^m)_{(n)}\prod_{i=0}^{m-1}\frac{1}{H_{\lambda^i}(t^m)}\right). \tag{3.8}$$

Since each $H_{\lambda^i}(t^m)$ is a product of factors of the form $(1-t^l)$, the product in the right most bracket consists entirely of factors of the form $(1-t^l)$. Therefore

$$t^{b_{\{\underline{\lambda}\}}} = t^k \prod_{i=0}^{m-1} t^{n(\lambda^i)m}$$

and equation (3.7) becomes

$$P(L(\{\underline{\mu}\},\epsilon),t) = \frac{\dim(\{\underline{\mu}\},\epsilon)\prod_{i=0}^{m-1} H_{\lambda^i}(t^m)}{(1-t)^n \tilde{R}_{\{\underline{\lambda}\}}(t)}.$$

□

To contradict Lemma 3.4 and hence prove Proposition 3.3 we will show:

Lemma 3.7. *Let $d \neq 1$ and $W = G(m,d,n)$ with $W \neq G(2,2,3)$. Then there exists $(\{\underline{\mu}\},\epsilon) \in Irr(W)$ such that the right hand side of equation (3.6) is not an element of $\mathbb{C}[t]$.*

Proof. We consider the cases $n = 2,3$ and $n > 3$ separately. For $n > 3$ choose $(\{\underline{\mu}\},\epsilon)$ such that its dual representation is $\underline{\lambda} = (\lambda^0, \emptyset, \dots \emptyset)$, where $\lambda^0 = (2,2,1,1,\dots 1)$. Then

$$\tilde{R}(t) = R(t) = 1 + t^{pn} + t^{2pn} + \dots + t^{(d-1)pn} = \frac{1-t^{mn}}{1-t^{pn}}$$

and for this particular m-multipartition we have

$$\prod_i H_{\lambda^i}(t^m) = H_{\lambda^0}(t^m) = (1-t^{2m})(1-t^m)(1-t^{(n-1)m})(1-t^{(n-2)m})\prod_{i=1}^{n-4}(1-t^{im}).$$

Equation (3.6) becomes

$$P(L(\{\underline{\mu}\},\epsilon),t) = \frac{\dim(\{\underline{\mu}\},\epsilon)(1-t^{2m})(1-t^m)(1-t^{(n-1)m})(1-t^{(n-2)m})(t^m)_{n-4}(1-t^{pn})}{(1-t^{mn})(1-t)^n}. \tag{3.9}$$

The numerator of (3.9) factorizes over $\mathbb{C}$ as a product of factors $(1-\omega t)$, where ω is a primitive k^{th} root of unity with $1 \leq k < mn$, whereas the denominator contains at least one factor of the form $(1-\sigma t)$, where σ is a primitive $(mn)^{th}$ root of unity. Therefore, since $\mathbb{C}[t]$ is an Euclidean domain, the right hand side of (3.9) cannot not lie in $\mathbb{C}[t]$. For $n = 2$ and $m \geq n$, take $\underline{\lambda} = ((1),(1),\emptyset \dots \emptyset)$. Then

$$\prod_i H_{\lambda^i}(t^m) = (1-t^m)^2 \qquad R(t) = \frac{t(1-t^{2m})}{1-t^{2p}}, \quad \text{and} \quad \tilde{R}(t) = \frac{1-t^{2m}}{1-t^{2p}}.$$

Substituting into (3.6)

$$P(L(\{\underline{\mu}\},\epsilon),t) = \frac{\dim(\{\underline{\mu}\},\epsilon)(1-t^m)^2(1-t^{2p})}{(1-t^{2m})(1-t)^2}.$$

By the same reasoning as above, since $2m > 2p$ and m, this rational function is not a polynomial. Similarly, for $n = 3$ and $m \geq n$, take $\underline{\lambda} = ((1),(1),(1),\emptyset \dots \emptyset)$. Then

$$\prod_i H_{\lambda^i}(t^m) = (1-t^m)^3 \qquad R(t) = \frac{t^3(1-t^{3m})}{1-t^{3p}}, \quad \text{and} \quad \tilde{R}(t) = \frac{1-t^{3m}}{1-t^{3p}}.$$

Substituting into (3.6)

$$P(L(\{\underline{\mu}\}, \epsilon), t) = \frac{\dim(\{\underline{\mu}\}, \epsilon)(1 - t^m)^3(1 - t^{3p})}{(1 - t^{3m})(1 - t)^3}.$$

Once again, this rational function is not a polynomial because $3m > 3p$ and m. $\square$

3.5 The Exceptional Groups

Using the computer algebra program GAP [86] together with the package CHEVIE [40] we calculated for each exceptional complex reflection group W (excluding G_4), the number of irreducible representations λ for which the polynomial $t^{-b_{\lambda^*}} f_{\lambda^*}(t)$ does not divide $P(\mathbb{C}[\mathfrak{h}]^{coW}, t)$ in $\mathbb{C}[t]$. Table (3.5) gives the results of these calculations. For each of these λ, Lemma 3.4 does not hold and hence $\dim L(\lambda) < |W|$ for all values of $\mathbf{c}$. Since there is always at least one such λ for every exceptional group, Proposition 3.3 is proved for the exceptional groups.

Table 3.1: Number of irreducibles that fail Lemma 3.4

Group	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
# failures	3	6	13	2	16	15	43	1	4	9	18	15	55	70	164	18	42

Group	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37
# failures	12	4	8	3	10	26	5	24	24	40	33	30	148	9	30	75

The code used to produce the data in Table (3.5) is given in Appendix A.4.

3.6 The exceptional group G_4

The group G_4, as labeled in [88], is the binary tetrahedral group. It can be realized as a finite subgroup of the group of units in the quaternions,

$$G_4 = \{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\},$$

and has order 24. It is generated by the elements $s_1 = \frac{1}{2}(-1 + i + j - k)$ and $s_2 = \frac{1}{2}(-1 + i - j + k)$ and has presentation $G_4 = \langle s_1, s_2 | s_1^3 = s_2^3 = (s_1 s_2)^6 = 1\rangle$. It has seven conjugacy classes which we label

$\mathrm{Cl}_1 = \{1\}$, Cl_2, Cl_3, Cl_4, Cl_5, Cl_6, and Cl_7. The character table is

Class	1	2	3	4	5	6	7
Size	1	1	4	4	6	4	4
Order	1	1	3	3	4	6	6
T	1	1	1	1	1	1	1
V_1	1	1	ω^2	ω	1	ω^2	ω
V_2	1	1	ω	ω^2	1	ω	ω^2
W	2	-2	-1	-1	0	1	1
$\mathfrak{h}$	2	-2	$-\omega^2$	$-\omega$	0	ω^2	ω
$\mathfrak{h}^*$	2	-2	$-\omega$	$-\omega^2$	0	ω	ω^2
U	3	3	0	0	-1	0	0

where ω is a primitive cube root of unity. Note that the reflection representation $\mathfrak{h}$ has dimension two, therefore G_4 is a rank two complex reflection group. The group G_4 has two conjugacy classes which consist of complex reflections and we label these reflections as

$$\mathrm{Cl}_3 = \{s_1, s_2, s_3, s_4\}$$

$$= \{\frac{1}{2}(-1+i+j-k), \frac{1}{2}(-1+i-j+k), \frac{1}{2}(-1-i+j+k), \frac{1}{2}(-1-i-j-k)\}$$

and

$$\mathrm{Cl}_4 = \{t_1, t_2, t_3, t_4\}$$

$$= \{\frac{1}{2}(-1-i-j+k), \frac{1}{2}(-1+i-j-k), \frac{1}{2}(-1-i+j-k), \frac{1}{2}(-1+i+j+k)\}.$$

Amazingly, unlike all other exceptional irreducible complex reflection groups, the generalized Calogero-Moser space associated to G_4 is smooth for generic values of the deformation parameter. In order to prove this we will require a pair of Lemmata about rational Cherednik algebras. For now, let $(W, \mathfrak{h})$ be any irreducible complex reflection group. Let $\{s_1, \dots s_k\}$ be a conjugacy class consisting of complex reflections in W and ζ the eigenvalue of s_1 (and hence all s_i) not equal to 1 when thinking of W as a subgroup of $GL(\mathfrak{h})$. For $1 \le i \le k$, let ω_{s_i} be the restricted symplectic form on $\mathfrak{h} \times \mathfrak{h}^*$ as defined in (2.3). Let $\pi_{s_i} : \mathfrak{h} \times \mathfrak{h}^* \to \mathrm{Im}\,(1 - s_i)$ be the projection map along $\mathrm{Ker}(1 - s_i)$, so that $\omega_{s_i} = \omega \circ \pi_{s_i}$, and define $\Omega = \sum_{i=1}^k \omega_{s_i}$.

Lemma 3.8. *Let W, ω and Ω be as above. Then*

$$\Omega = \frac{k}{n}(1-\zeta)^{-1}(1-\zeta^{-1})^{-1}(2-\zeta-\zeta^{-1})\omega.$$

Proof. Since each ω_{s_i} is alternating and $\mathbb{C}$-linear, $\Omega \in \bigwedge^2 (\mathfrak{h} \times \mathfrak{h}^*)$. Let $x \in \mathfrak{h} \times \mathfrak{h}^*$. Then x decomposes uniquely as $x_1 + x_2$, with $x_1 \in \mathrm{Im}\,(1-s_i)$ and $x_2 \in \mathrm{Ker}(1-s_i)$. By definition, there exists $y \in \mathfrak{h} \times \mathfrak{h}^*$ such that $(1-s_i)y = x_1$. Then $(1-gs_ig^{-1})(gy) = gx_1$ implies that $gx_1 \in \mathrm{Im}\,(1-gs_ig^{-1})$. Also $(1-s_i)x_2 = 0$ implies that $(1-gs_ig^{-1})gx_2 = 0$ hence gx decomposes as $gx_1 + gx_2$ with $gx_1 \in \mathrm{Im}\ (1-gs_ig^{-1})$ and

$gx_2 \in \mathrm{Ker}\,(1 - gs_ig^{-1})$. Therefore $\pi_{gs_ig^{-1}} = g\pi_{s_i}g^{-1}$ and $\omega_{s_i}(g^{-1}x, g^{-1}y) = \omega_{gs_ig^{-1}}(x,y)$. Hence $\Omega \in \left(\bigwedge^2(\mathfrak{h}^* \times \mathfrak{h})\right)^W$. Lemma 1.18 says that $\dim \left(\bigwedge^2(\mathfrak{h}^* \times \mathfrak{h})\right)^W = 1$, therefore there exists $\lambda \in \mathbb{C}$ such that $\Omega = \lambda\omega$. Consider $\Omega'(x,y) = \Omega((x,0),(0,y))$, where $x \in \mathfrak{h}$ and $y \in \mathfrak{h}^*$. Recall that ζ is the eigenvalue of s_i not equal to 1, then $\pi_{s_i}(x) = (1-\zeta)^{-1}(1-s_i)x$ and $\pi_{s_i}(y) = (1-\zeta^{-1})^{-1}(1-s_i)y$. Expanding $\Omega'(x,y)$:

$$\Omega'(x,y) = \sum_{i=1}^{k} \omega((1-\zeta)^{-1}(1-s_i)x, (1-\zeta^{-1})^{-1}(1-s_i)y)$$

$$= (1-\zeta)^{-1}(1-\zeta^{-1})^{-1}\sum_{i=1}^{k}[\omega(x,y) - \omega(s_ix,y) - \omega(x,s_iy) + \omega(s_ix,s_iy)]$$

$$= (1-\zeta)^{-1}(1-\zeta^{-1})^{-1}\omega(x, (\sum_{i=1}^{k} 2 - s_i - s_i^{-1})y).$$

Define $\phi = (\sum_{i=1}^{k} 2 - s_i - s_i^{-1}) : \mathfrak{h}^* \to \mathfrak{h}^*$, a W-homomorphism. The trace of ϕ is $2nk - (n-1)k - k\zeta - (n-1)k - k\zeta^{-1} = k(2 - \zeta - \zeta^{-1})$. Since $\mathfrak{h}^*$ is irreducible, Schur's lemma says that $\phi(y) = \frac{k}{n}(2-\zeta-\zeta^{-1})y$ and hence $\lambda = \frac{k}{n}(1-\zeta)^{-1}(1-\zeta^{-1})^{-1}(2-\zeta-\zeta^{-1})$. $\square$

We also require the notion of a generalized baby Verma module, which are baby Verma modules above points other than the origin in $\mathfrak{h}/W \times \mathfrak{h}^*/W$.

Definition 3.9. Let $(p,q) \in \mathfrak{h}/W \times \mathfrak{h}^*$, W_q be the stabilizer subgroup of q in W and E be an irreducible W_q-module. Then we define the *generalized baby Verma* module

$$\Delta_{\mathbf{c}}(E;p,q) := H_{0,\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*] \rtimes W_q} E,$$

where the action of $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*] \rtimes W_q$ on E is given by $(f \otimes g \otimes w) \cdot e = f(p)g(q)w \cdot e$ for all $f \in \mathbb{C}[\mathfrak{h}]^W$, $g \in \mathbb{C}[\mathfrak{h}^*]$, $w \in W_q$ and $e \in E$.

Since $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W \subseteq Z_{\mathbf{c}}$, Schur's lemma implies that, for every irreducible $H_{0,\mathbf{c}}(W)$-module L, there exists $(p,r) \in \mathfrak{h}/W \times \mathfrak{h}^*/W$ such that $(f \otimes g) \cdot l = f(p)g(r)l$, for all $l \in L$, $f, g \in \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$. Choosing a point q in the orbit represented by r we write $(p,r) = (p, Wq)$ and say that the irreducible $H_{0,\mathbf{c}}$-module L *lies above* (p, Wq).

Lemma 3.10. *Let L be an irreducible $H_{0,\mathbf{c}}(W)$-module lying above (p,Wq). Then there exist $E \in \mathsf{Irr}\,(W_q)$ and a surjective $H_{0,\mathbf{c}}(W)$-homomorphism $\phi : \Delta_{\mathbf{c}}(E;p,q) \twoheadrightarrow L$.*

Proof. The action on L of the commutative ring $\mathbb{C}[\mathfrak{h}^*]$ gives a decomposition $L = \oplus_{q' \in \mathfrak{h}^*} L_{q'}^{\mathrm{gen}}$ of L into generalized eigenspaces. That is, for each $l \in L_{q'}^{\mathrm{gen}}$ and $f \in \mathbb{C}[\mathfrak{h}^*]$, there exists an $N \in \mathbb{N}$ such that $(f - f(q'))^N \cdot l = 0$ (since L is finite dimensional, we can choose N to be independent of f and l). Choose q' such that $L_{q'}^{\mathrm{gen}} \neq 0$, so that $(f - f(q'))^N$ acts as zero on $L_{q'}^{\mathrm{gen}}$ for all $f \in \mathbb{C}[\mathfrak{h}^*]^W$. As L lies over (p,Wq) we see that $(f - f(q))$ also acts nilpotently on $L_{q'}^{\mathrm{gen}}$ and $f(q) = f(q')$. Since W is a finite group, each orbit in $\mathfrak{h}^*$ is closed, therefore $q' \in Wq$ and we can find $w \in W$ such that $w \cdot q = q'$. Now let $0 \neq L_{q'} \subseteq L_{q'}^{\mathrm{gen}}$ be the space of elements l in $L_{q'}^{\mathrm{gen}}$ such that $(f - f(q')) \cdot l = 0$, for all $f \in \mathbb{C}[\mathfrak{h}^*]$. Then

$w^{-1}(L_{q'}) \neq 0$ and $f \cdot (w^{-1}l) = w^{-1} \cdot ({}^w f)(q')l = f(q)w^{-1} \cdot l$ implies that $w^{-1}(L_{q'}) \subseteq L_q$. Thus L_q is a nonzero W_q-submodule of L because $f \cdot (v \cdot l) = v \cdot f(q)l = f(q)(v \cdot l)$ for all $f \in \mathbb{C}[\mathfrak{h}]$, $v \in W_q$ and $l \in L_q$. Choose an irreducible W_q-submodule E of L_q. The inclusion $E \hookrightarrow L$ induces a $H_{0,\mathbf{c}}(W)$-homomorphism $\phi : \Delta_{\mathbf{c}}(E; p, q) \to L$. The fact that L is irreducible implies that this is a surjection. $\square$

Theorem 3.11. *For generic values of* $\mathbf{c}$*, the generalized Calogero-Moser space* $X_{\mathbf{c}}$ *associated to* G_4 *is a smooth variety.*

Proof. The theorem is proved by showing that each irreducible $H_{0,\mathbf{c}}(G_4)$-module is isomorphic to the regular representation of G_4. By Proposition 3.2, this is equivalent to the statement of the theorem. Let $E = T \oplus V_1 \oplus V_2 \oplus 3U$ and $F = \mathfrak{h} \oplus \mathfrak{h}^* \oplus W$ be two G_4-modules.

Claim 1

Let L be a finite dimensional $H_{0,\mathbf{c}}(G_4)$-module for $\mathbf{c}$ generic, then $L \cong aE \oplus bF$, for some $a, b \in \mathbb{Z}_{\geq 0}$.

To prove Claim 1 we use an argument similar to that of [36, Proposition 16.5]. Let $\rho : H_{0,\mathbf{c}} \to \mathrm{End}_{\mathbb{C}}(L)$ realize the action of $H_{0,\mathbf{c}}$ on L. Then, for all $x, y \in \mathfrak{h} \oplus \mathfrak{h}^*$, we have the commutation relation

$$[\rho(x), \rho(y)] = c_1 \sum_{i=1}^{4} \omega_{s_i}(x, y)\, \rho(s_i) + c_2 \sum_{j=1}^{4} \omega_{t_j}(x, y)\, \rho(t_j) \tag{3.10}$$

By Lemma 3.8, $\sum_{i=1}^{4} \omega_{s_i} = \sum_{j=1}^{4} \omega_{t_j} = 2\omega$. Taking traces on both sides of equation (3.10) gives

$$0 = c_1\, 2\omega(x, y)\, \mathrm{Tr}_L(s_1) + c_2\, 2\omega(x, y)\, \mathrm{Tr}_L(t_1) \qquad \forall x, y \in \mathfrak{h} \oplus \mathfrak{h}^* \tag{3.11}$$

Since c_1 and c_2 are generic and equation (3.11) is linear, we have $0 = 2\omega(x, y)\, \mathrm{Tr}_L(s_1) = 2\omega(x, y)\, \mathrm{Tr}_L(t_1)$. The fact that ω is non-degenerate implies that Tr_L is zero on Cl_3 and Cl_4. Using the fact that s_1 is a complex reflection and $\dim \mathfrak{h}^*$ equals two, we can choose a non-zero $x_1 \in \mathfrak{h}^*$ such that $s_1(x_1) = x_1$. Then $s_1[x_1, y] = [x_1, s_1 y]$ for all $y \in \mathfrak{h}$. Since $s_1(x_1) = x_1$, $x_1 \in \mathrm{Ker}\ (1 - s_1)$ and hence $\omega_{s_1}(x_1, y) = 0$ for all $y \in \mathfrak{h}$. Similarly, $s_1 t_1 = 1$ implies that $x_1 \in \mathsf{Fix}(t_1)$ and hence $\omega_{t_1}(x_1, y) = 0$. Therefore, multiplying both sides of equation (3.10) on the left by $\rho(s_1)$ and taking traces gives

$$0 = c_1 \sum_{i=2}^{4} \omega_{s_i}(x_1, y)\, \mathrm{Tr}_L(s_1 s_i) + c_2 \sum_{j=2}^{4} \omega_{t_j}(x_1, y)\, \mathrm{Tr}_L(s_1 t_j).$$

Again, the fact that c_1, c_2 are generic, implies that

$$0 = \sum_{i=2}^{4} \omega_{s_i}(x_1, y)\, \mathrm{Tr}_L(s_1 s_i) = \sum_{j=2}^{4} \omega_{t_j}(x_1, y)\, \mathrm{Tr}_L(s_1 t_j).$$

Since s_1s_2, s_1s_3 and s_1s_4 all belong to Cl_7 and s_1t_2, s_1t_3, s_1t_4 all belong to Cl_5 we have

$$0 = \sum_{i=2}^{4} \omega_{s_i}(x_1, y)\,\mathrm{Tr}_L(s_1s_i) = 2\omega(x_1, y)\,\mathrm{Tr}_L(s_1s_2),$$

$$0 = \sum_{j=2}^{4} \omega_{t_j}(x_1, y)\,\mathrm{Tr}_L(s_1t_j) = 2\omega(x_1, y)\,\mathrm{Tr}_L(s_1t_2).$$

Therefore Tr_L is zero on Cl_7 and Cl_5. We can also multiplying both sides of equation (3.10) on the left by $\rho(t_1)$ instead of $\rho(s_1)$. Noting that $t_1^2 \in \mathrm{Cl}_3$, $t_1t_2, t_1t_3, t_1t_4 \in \mathrm{Cl}_6$ and repeating the above argument shows that Tr_L is also zero on Cl_6. Therefore any element of G_4 that has non-zero trace on L must belong to Cl_1 or Cl_2. Hence the character associated to L must take values $(n, m, 0, 0, 0, 0, 0)$, for some $n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}$, on the conjugacy classes $\mathrm{Cl}_1, \mathrm{Cl}_2, \dots, \mathrm{Cl}_7$. Taking inner products shows that

$$L \cong \frac{1}{|G_4|}(n+m)\,E \oplus \frac{2}{|G_4|}(n-m)\,F.$$

Setting $a = \frac{1}{|G_4|}(n+m)$ and $b = \frac{2}{|G_4|}(n-m)$ proves Claim 1.

Claim 2

Let L be an irreducible representation of $H_{0,\mathbf{c}}(G_4)$, with $\mathbf{c}$ generic. Then L must be isomorphic to $E \oplus F$ or $\mathbb{C}\,G_4$ as a G_4-module.

If L is irreducible then $\dim L \leq 24$. Therefore Claim 1 implies that $L \cong E, 2E, nF, 1 \leq n \leq 4, E \oplus F$ or $\mathbb{C}G_4$. Assume that L is isomorphic to E as a G_4-module. The action of $\mathfrak{h}^*$ on L defines a G_4-equivariant linear map $\phi : \mathfrak{h}^* \to \mathrm{End}_{\mathbb{C}}(E)$. The G_4-module $\mathrm{End}_{\mathbb{C}}(E)$ decomposes as

$$\mathrm{End}_{\mathbb{C}}(E) \cong (T \otimes T) \oplus 2(T \otimes V_1) \oplus 2(T \otimes V_2) \oplus 6(T \otimes U) \oplus (V_1 \otimes V_1) \oplus 2(V_1 \otimes V_2) \oplus$$

$$6(V_1 \otimes U) \oplus (V_2 \otimes V_2) \oplus 6(V_2 \otimes U) \oplus 9(U \otimes U) \cong 12T \oplus 12V_1 \oplus 12V_2 \oplus 36U$$

This shows that $\mathfrak{h}^*$ is not a summand of $\mathrm{End}_{\mathbb{C}}(E)$. Thus ϕ must be the zero map. Similarly, the action of $\mathfrak{h}$ must also be zero on E. This implies that the right hand side of equation (3.10) must also act as zero on E. In particular, it must act as zero on $T \subset E$. This means that

$$0 = c_1 \sum_{i=1}^{4} \omega_{s_i}(x, y) + c_2 \sum_{j=1}^{4} \omega_{t_j}(x, y) = 2(c_1 + c_2)\omega(x, y)$$

This is a contradiction because c_1, c_2 are generic and ω is non-degenerate. Hence L cannot be isomorphic to E. Repeating the above argument for F we have

$$\mathrm{End}_{\mathbb{C}}(F) \cong (\mathfrak{h} \otimes \mathfrak{h}) \oplus 2(\mathfrak{h} \otimes \mathfrak{h}^*) \oplus 2(\mathfrak{h} \otimes W) \oplus$$

$$(\mathfrak{h}^* \otimes \mathfrak{h}^*) \oplus 2(\mathfrak{h}^* \otimes W) \oplus (W \otimes W) \cong 3T \oplus 3V_1 \oplus 3V_2 \oplus 9U$$

Therefore $\mathfrak{h}^*$ and $\mathfrak{h}$ must act as zero on F. If we consider the right hand side of equation (3.10), this time restricted to $W \subset F$ then we have

$$0 = c_1 \sum_{i=1}^{4} \omega_{s_i}(x,y)\, \rho|_W(s_i) + c_2 \sum_{j=1}^{4} \omega_{t_j}(x,y)\, \rho|_W(t_j)$$

Taking the trace of this equation gives $0 = -2(c_1 + c_2)\omega(x,y)$, which is a contradiction because c_1, c_2 are generic and ω is non-degenerate. Therefore $L \ncong F$. The same reasoning shows that L cannot be isomorphic to $2E$ or $nF, 2 \leq n \leq 4$ either. This proves Claim 2.

Claim 3

Let L be an irreducible $H_{0,\mathbf{c}}(G_4)$-module, then L cannot be isomorphic to $E \oplus F$ as a G_4-module.

By Lemma 3.10, there exists a generalized Verma module $\Delta_{\mathbf{c}}(M; p, q)$ and a surjective homomorphism $\phi : \Delta_{\mathbf{c}}(M; p, q) \twoheadrightarrow L$. As a G_4-module we have

$$\Delta_{\mathbf{c}}(M; p, q) = H_{0,\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*] \rtimes W_q} M \cong \mathbb{C}G_4 \otimes \mathrm{Ind}_{(G_4)_q}^{G_4} M \cong k\mathbb{C}G_4$$

where $(G_4)_q$ is the stabilizer of $q \in \mathfrak{h}^*$ and $k = [G_4 : (G_4)_q] \dim M$. The generalized Verma module $\Delta_{\mathbf{c}}(M; p, q)$ has a finite composition series. Each factor of this series must have dimension ≤ 24. Therefore, by Claim 2, each factor is isomorphic to either $\mathbb{C}G_4$ or $E \oplus F$ as a G_4-module. Hence there exist $m, n \in \mathbb{N}$ such that $k\mathbb{C}G_4 \cong m\mathbb{C}G_4 \oplus n(E \oplus F)$ with $n \geq 1$. But then $n(E \oplus F) \cong (k - m)\mathbb{C}G_4$, which is a contradiction. This completes the proof of Claim 3 and the theorem. $\square$

3.7 Remarks

1. The results of this chapter have been published in the article [7].
2. Proposition 3.2 is still valid for symplectic reflection algebras. However there does not seem to be (as yet) any good method of producing simple modules. The main reasons for this are that there is no well understood central subalgebra of the symplectic reflection algebra and the algebra is no longer $\mathbb{Z}$-graded. In particular, there is no analogue of the restricted rational Cherednik algebra.

Chapter 4

Symplectic resolutions of quotient singularities

In this chapter we investigate an important question in symplectic algebraic geometry, that of the existence of symplectic resolutions for symplectic singularities. In particular we examine whether these symplectic resolutions exist for the quotient of a symplectic vector space by a finite group acting by symplectomorphisms. Results of Ginzburg-Kaledin and Namikawa show that answering this question, at least for symplectic resolutions that are projective over their bases, is equivalent to classifying those quotient varieties whose corresponding generalized Calogero-Moser space is smooth for generic values of the deformation parameter. Therefore the main result of the previous chapter provides a classification theorem for the existence of symplectic resolutions, projective over their bases, for a large class of examples of quotient symplectic singularities.

4.1 Symplectic singularities

Throughout this chapter, a variety will mean an integral scheme of finite type over $\mathbb{C}$. In the paper [3], Beauville introduces the notion of symplectic singularities, based on the notion of rational Gorenstein singularities.

Definition 4.1. Let X be a normal, even dimensional variety and assume that there exists a non-degenerate symplectic 2-form ω on the smooth locus X_{sm} of X. The variety X is said to have *symplectic singularities* (or the pair (X, ω) is said to be a *symplectic variety*) if there exists a resolution $\pi : Y \to X$ of singularities such that the pull-back of ω to $\pi^{-1}(X_{\mathrm{sm}})$ extends to a regular 2-form on the whole of Y.

As noted in [3], in the above definition it suffices to show that there exists one resolution Y of X such that the pull-back of ω to $\pi^{-1}(X_{\mathrm{sm}})$ extends to a 2-form on the whole of Y.

Definition 4.2. Let (X, ω) be a variety with symplectic singularities. The resolution $\pi : Y \to X$ is said to be a *symplectic resolution* if the extension of $\pi^*\omega$ is a non-degenerate closed two-form on Y.

Example 4.3. Let G be a simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Fix B some Borel subgroup of G with Lie algebra $\mathfrak{b}$ and let G/B be the flag manifold of G. The tangent space to G/B at $g \cdot B$ is $\mathfrak{g}/Ad_g(\mathfrak{b})$, hence the cotangent space to G/B at $g \cdot B$ is $\{\lambda \in \mathfrak{g}^* \mid \lambda(Ad_g(\mathfrak{b})) = 0\} = (Ad_g(\mathfrak{b}))^{\perp}$. Therefore the image of the natural map $\pi : T^*(G/B) \to \mathfrak{g}^*$ is the nilpotent cone $\mathcal{N}$, as introduced in (1.11). It was shown by Springer [91] that $\pi : T^*(G/B) \to \mathcal{N}$ is a resolution of the singular Poisson variety $\mathcal{N}$. Let $\mathcal{O} \subset \mathcal{N}$ be a nilpotent orbit. The closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ is a Poisson subvariety of $\mathcal{N}$. However it is not usually a normal variety. We denote the normalization of $\overline{\mathcal{O}}$ by $\widetilde{\mathcal{O}}$. Panyushev [80] showed that each space $\widetilde{\mathcal{O}}$ has symplectic singularities. Then Fu [38, Theorem 0.1] showed that if there exists a symplectic resolution of $\widetilde{\mathcal{O}}$ then it must be of the following form: there exists a parabolic subgroup P of G such that the map

$$\pi : T^*(G/P) \simeq G \times_P \mathfrak{p}^{\perp} \to \widetilde{O}, \quad (g, X) \mapsto Ad_g^*(X)$$

is a symplectic resolution of $\widetilde{O}$, where $\mathfrak{p}$ the Lie algebra of P.

Definition 4.4. A morphism $\pi : Y \to X$ is called *semi-small* if for every closed subvariety F in Y we have

$$2 \cdot \mathrm{codim}_Y\, F \geq \mathrm{codim}_X\, \pi(F).$$

To demand that a resolution of singularities is semi-small puts a large restriction on the resolutions that one may consider. It was shown by Kaledin that symplectic resolutions are always semi-small hence symplectic resolutions are very “special”.

Proposition 4.5 (Proposition 1.2,[63])**.** *Suppose that (X, ω) has symplectic singularities and $\pi : Y \to X$ is a symplectic resolution. Then π is semi-small.*

The main class of examples of symplectic singularities that will be of interest to us are those arising as the quotient of a symplectic vector space V by a finite subgroup of $Sp(V)$.

Proposition 4.6 ([3], Proposition 2.4)**.** *Let V be a symplectic vector space and $G \subset Sp(V)$ a finite group. Then the quotient variety V/G is has symplectic singularities.*

Since $Sp(V) \subseteq SL(V)$, the Chevalley-Shephard-Todd Theorem (1.12) implies that the variety V/G will always be singular if $G \neq 1$. Therefore it is natural to ask:

Q. For which finite groups $G \subset Sp(V)$ does there exist a symplectic resolution, projective over its base, of V/G?

Analogous to the converse proved by Shephard and Todd to Chevalley's Theorem, Verbitsky [99] proved:

Theorem 4.7. *Suppose that V/G admits a symplectic resolution, then G is a symplectic reflection group.*

However, as will be shown below, the result that would be analogous to Chevalley's Theorem is not true - there exist symplectic reflection groups G such that V/G does not admit a symplectic resolution, projective over its base. In fact we will show that very few symplectic reflection groups have this property.

4.2 The Poisson deformation functor

Let $(X, \{\ ,\ \})$ be a Poisson scheme over $\mathbb{C}$ and S a local Artinian $\mathbb{C}$-algebra with $S/\mathfrak{m}_S = \mathbb{C}$. Fix $T = \operatorname{Spec} S$.

Definition 4.8. A Poisson deformation of $(X, \{\ ,\ \})$ over S is a Poisson T-scheme $(\mathcal{X}, \{\ ,\ \}_T)$, flat over T, together with a Poisson isomorphism $\phi : \mathcal{X} \times_T \operatorname{Spec}(\mathbb{C}) \to X$. We say that the deformations $(\mathcal{X}, \{\ ,\ \}_T)$ and $(\mathcal{X}', \{\ ,\ \}_T)$ are equivalent if there exists a Poisson isomorphism $\psi : \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ such that $\phi' \circ \psi = \phi$.

Let $\mathbb{PD}_{(X,\{,\})}(S)$ denote the set of all equivalence classes of Poisson deformations of $(X, \{\ ,\ \})$ over S. This defines a functor $\mathbb{PD}_{(X,\{,\})} : (\mathrm{Art})_{\mathbb{C}} \to (\mathrm{Set})$ from the category of local Artinian $\mathbb{C}$-algebras to the category of sets. In this situation we say that $\mathbb{PD}$ is *pro-representable* if there exists some complete, local $\mathbb{C}$-algebra R with maximal ideal $\mathfrak{m}$ such that $R/\mathfrak{m}^n \in (\mathrm{Art})_{\mathbb{C}}$ for all $n \geq 1$ and there is an equivalence of functors $\mathbb{PD}_{(X,\{,\})} \simeq \mathrm{Hom}_{\mathrm{local}}(R, -)$. This setup fits into the formalism of functors of Artinian rings as developed by Schlessinger. In the article [87], Schlessinger states necessary and sufficient conditions for a functor $F : (\mathrm{Art})_{\mathbb{C}} \to (\mathrm{Set})$ to be pro-representable. In the papers [79] and [78] Namikawa studies Poisson deformations using the theory of Poisson cohomology. The outcome of this work is the deep and difficult result:

Theorem 4.9 (Theorem 2.7, [78])**.** *Let (X, ω) be an affine variety with symplectic singularities, then the functor $\mathbb{PD}_{(X,\{,\})}$ is pro-representable.*

We will denote the complete, local algebra guaranteed by Theorem 4.9 by R_X so that $\mathbb{PD}_{(X,\{,\})}(-) \simeq \mathrm{Hom}_{local}(R_X, -)$.

4.3 Deformations vs. Resolutions

The Poisson deformations described in section 4.2 were all local. However in specific examples the deformations that one can explicitly construct are not local in nature. One could overcome this by simply completing at the special fiber but then we lose information that might be gleamed by studying other fibers of the deformation. Another way around this problem is to consider instead local, *graded* Poisson deformations.

Definition 4.10. Let (X, ω) be an affine variety with symplectic singularities, equipped with a $\mathbb{C}^*$-action. We say that (X, ω) has a *good* $\mathbb{C}^*$*-action* if:

1. with respect to the $\mathbb{C}^*$-action, the coordinate ring $\mathbb{C}[X]$ is $\mathbb{N}$-graded and $\mathbb{C}[X]_0 = \mathbb{C}$ (this implies that there exists a unique fixed point $o \in X$),

2. the symplectic form ω on X_{sm} has positive weight $l > 0$ with respect to $\mathbb{C}^*$.

Here ω is said to have weight l with respect to $\mathbb{C}^*$ if $\lambda \cdot \omega = \lambda^l \omega$, for all $\lambda \in \mathbb{C}^*$. The action of $\mathbb{C}^*$ on Ω_X^2 is explicitly given as follows. Let ζ a vector field on X. Then $(\lambda \cdot \zeta)(f) := \zeta(\lambda^{-1} \cdot f)$ for all $f \in \mathbb{C}[X]$. Now if ζ_1 and ζ_2 are two vector fields on X then $(\lambda \cdot \omega)(\zeta_1, \zeta_2) := \omega(\lambda^{-1} \cdot \zeta_1, \lambda^{-1} \cdot \zeta_2)$.

If (X, ω) is an affine variety with symplectic singularities, equipped with a $\mathbb{C}^*$-action, then the symplectic form ω on X_{sm} has weight l with respect to $\mathbb{C}^*$ if and only if the corresponding Poisson bracket on $\mathbb{C}[X]$ is graded of degree $-l$.

Proposition 4.11 (page 19, [78])**.** *Let (X, ω) be an affine variety with symplectic singularities, equipped with a good $\mathbb{C}^*$-action. Then the complete, local ring R_X is $\mathbb{N}$-graded and the universal Poisson deformation $\mathcal{X} \to Spec(R_X)$ is $\mathbb{C}^*$-equivariant.*

As shown in [79, Lemma (A.2)] the space R_0 of all eigenvectors of the $\mathbb{C}^*$-action is a finitely generated $\mathbb{C}$-algebra with good $\mathbb{C}^*$-action. In fact the proof of [78, Theorem 2.9] shows that $\mathrm{Spec}\,(R_0) \simeq \mathbb{A}^n$ for some n. A consequence of this, which is of fundamental importance for us, is the following result of Namikawa's.

Theorem 4.12 (Corollary 2.10, [78])**.** *Let (X, ω) be an affine variety with symplectic singularities, equipped with a good $\mathbb{C}^*$-action, then the following are equivalent:*

1. *X has a symplectic resolution, projective over its base.*
2. *X has a smoothing by a Poisson deformation.*

The symplectic reflection algebras provide us with a "canonical" flat family of varieties $X_{\mathbf{c}}(G) := \mathrm{Spec} Z(H_{0,\mathbf{c}}(G))$ such that $V/G = X_0(G)$. This family actually defines a Poisson deformation of V/G. It is shown in [50, Proposition 4.5] that each fiber has symplectic singularities. We repeat below the proof given there.

Proposition 4.13. *The flat family $\{X_{\mathbf{c}}(G) \,|\, \mathbf{c} \in \mathbb{C}[\mathcal{S}]^G\}$ defines a Poisson deformation of $V/G = X_0(G)$. Moreover, each fiber $X_{\mathbf{c}}(G)$ has symplectic singularities.*

Proof. It is shown by Gordon and Smith, [52, Lemma 3.10] that the symplectic reflection algebra $H_{0,\mathbf{c}}(G)$ defines a noncommutative resolution of V/G in the sense of Van den Bergh [97]. Therefore, by [92, Theorem 4.3], the variety $X_{\mathbf{c}}(G)$ has rational singularities. Moreover it is shown in [17, Theorem 7.8] that the Poisson structure of $X_{\mathbf{c}}(G)$ is non-degenerate when restricted to the smooth locus. Now [77, Theorem 6] implies that $X_{\mathbf{c}}(G)$ has symplectic singularities. The fact that the family is flat is a consequence of the Satake isomorphism, Theorem 2.5, and the PBW theorem for symplectic reflection algebras. Finally, the fact that the deformation is Poisson follows from the fact that one can define the Poisson bracket, as in (1.3), on the total space of the deformation. □

Lemma 4.14. *Let (V, ω, G) be an indecomposable triple. The variety V/G has a good $\mathbb{C}^*$-action and the generalized Calogero-Moser deformation $\phi : X_{\mathbf{c}}(G) \to \mathcal{S}/G$ is a graded deformation of V/G.*

Proof. Equip V with the $\mathbb{C}^*$-action given by $\lambda \cdot v = \lambda^{-1} v$, where $\lambda \in \mathbb{C}^*$ and $v \in V$. Then the weights of this $\mathbb{C}^*$-action on $\mathbb{C}[V]$ are all positive. The action of G commutes with the action of $\mathbb{C}^*$ on $\mathbb{C}[V]$, therefore $\mathbb{C}[V]^G$ is a positively graded subalgebra of $\mathbb{C}[V]$. The ideal $\mathbb{C}[V]^G_+$ of functions with constant term zero is a maximal ideal in $C[V]^G$ and defines the unique fixed point $o \in V/G$ of the $\mathbb{C}^*$-action on V/G. Lemma 1.19 says that the Poisson bracket on $\mathbb{C}[V]^G$ corresponding to ω has degree -2. Therefore the corresponding 2-form on $(V/G)_{\mathrm{sm}}$ will have weight two. To make the generalized Calogero-Moser deformation graded one should define a $\mathbb{N}$-grading on $\mathbb{C}[\mathcal{S}]^G$ such that the characteristic functions that take the value one on a particular conjugacy class in $\mathcal{S}$ and zero on all others has degree two. Then it follows from the defining relations (2.1), see also the proof of Theorem 2.3, that the total space of the deformation inherits a $\mathbb{C}^*$-action such that the map ϕ is equivariant. $\square$

Let $\widehat{\mathcal{S}}/G := \mathrm{Spec}\,(\widehat{\mathbb{C}[\mathcal{S}/G]})$, where $\widehat{\mathbb{C}[\mathcal{S}/G]}$ is the completion of $\mathbb{C}[\mathcal{S}/G]$ with respect to the augmentation ideal. By definition of the universal Poisson deformation, there exists a classifying map $\kappa : \widehat{\mathcal{S}}/G \to \mathrm{Spec}\,(R_{V/G})$. Since the generalized Calogero-Moser deformation is $\mathbb{N}$-graded this lifts to an equivariant classifying map $\kappa : \mathcal{S}/G \to \mathrm{Spec}\,(R_0) = \mathbb{A}^n$ such that the generalized Calogero-Moser deformation is realized as the Cartesian product

$$\begin{tikzcd} X_{\mathbf{c}}(G) \arrow[r] \arrow[d, "\phi"'] & \mathcal{X} \arrow[d] \\ \mathcal{S}/G \arrow[r, "\kappa"'] & \mathbb{A}^n \end{tikzcd}$$

The morphism κ is not, in general, an isomorphism but it is shown in [45, Proposition 1.16] that $n = \dim\, \mathcal{S}/G$. Although the generalized Calogero-Moser deformation is not the universal graded Poisson deformation, Ginzburg and Kaledin [45] have shown that the generalized Calogero-Moser deformation "sees" all possible deformations when there exists a symplectic resolution, projective over its base, for V/G:

Theorem 4.15 (Theorem 1.20 and Proposition 1.18, [45])**.** *Assume that there exists a symplectic resolution, projective over its base, $Y \to V/G$, then the classifying map κ is surjective and generically étale. For a generic parameter $\mathbf{c} \in \mathcal{S}/G$ the natural map $X_{\mathbf{c}}(G) \to \mathcal{X}_{\kappa(\mathbf{c})}$ is an isomorphism.*

Combining Theorems 4.12 and 4.15.

Corollary 4.16. *Let (V, ω, G) be an indecomposable triple. There exists a symplectic resolution, projective over its base, for V/G if and only if the generalized Calogero-Moser deformation $X_{\mathbf{c}}(G)$ is smooth for generic values of the parameter $\mathbf{c}$.*

4.4 Classification

We can now state the classification theorem for the symplectic singularities $\mathfrak{h} \times \mathfrak{h}^*/W$, where W is an irreducible complex reflection group. Let Γ be a finite subgroup of $SL_2(\mathbb{C})$[1]. By describing the generalized Calogero-Moser space associated to the wreath product $\Gamma \wr S_n$ as an affine quiver variety, Etingof and Ginzburg [36, Corollary 1.14] have shown that, for generic values of the parameter $\mathbf{c}$, $X_{\mathbf{c}}(S_n \wr \Gamma)$ is a smooth variety. Similarly, we have shown in Theorem 3.11 that the generalized Calogero-Moser space associated to the complex reflection group G_4 is smooth for generic values of the deformation parameter. Therefore we can apply Namikawa's Theorem 4.12 to deduce that:

[1]Of course when Γ is a finite subgroup of $SL_2(\mathbb{C})$ of type D or E the space $\mathbb{C}^{2n}/(\Gamma \wr S_n)$ cannot be described as $\mathfrak{h} \times \mathfrak{h}^*/W$ for some complex reflection group W but the results of Etingof and Ginzburg apply in this more general setup.

Corollary 4.17. *Let Γ be a finite subgroup of $SL_2(\mathbb{C})$. There exist symplectic resolutions, projective over their bases, for the symplectic varieties $\mathbb{C}^{2n}/(\Gamma \wr S_n)$ and $\mathfrak{h} \times \mathfrak{h}^*/G_4$.*

Theorem 4.15 together with Theorem 3.1 imply:

Corollary 4.18. *Let W be an irreducible complex reflection group, $W \not\simeq G(m,1,n)$ or G_4. Then there does not exist a symplectic resolution, projective over its base, for the symplectic singularity $\mathfrak{h} \times \mathfrak{h}^*/W$.*

Cohen [27] has classified all indecomposable triples (V, ω, G). The classification can be summarized as follows: there are those indecomposable triples coming from doubling up an irreducible complex reflection group, certain normal subgroups of $\Gamma \wr S_n$ as listed [27, Theorem 2.9] and a finite list of exceptional groups whose corresponding symplectic vector spaces have dimensions ranging from four to ten. Therefore the following question is still not completely answered.

Q: For which indecomposable triples (V, ω, G) do there exist a symplectic resolution, projective over its base, for V/G?

As we have shown above this question is equivalent to classifying those triples whose corresponding generalized Calogero-Moser space is smooth for generic values of the deformation parameter, see (3.7).

4.5 Explicit resolutions

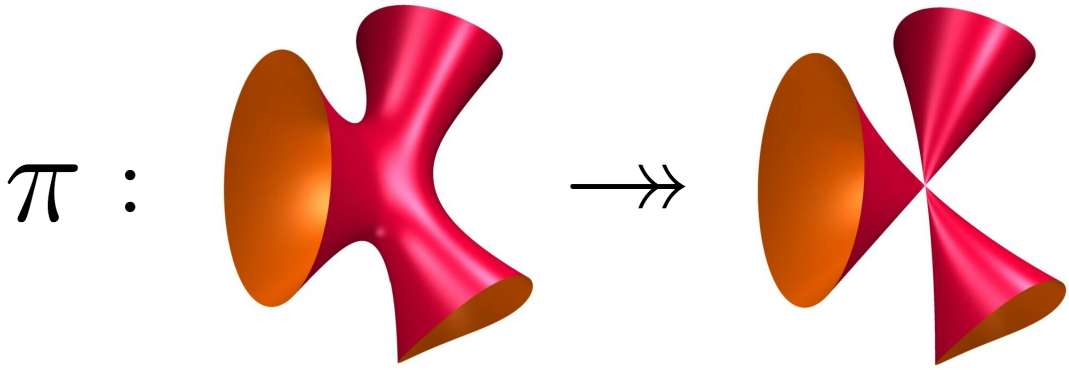
In [100] Wang constructs an explicit symplectic resolution of the symplectic singularity $\mathbb{C}^{2n}/\Gamma \wr S_n$, where Γ is a finite subgroup of $SL_2(\mathbb{C})$. We describe here his construction. For any quasi-projective, smooth variety X we denote by $\mathrm{Hilb}^n(X)$ the Hilbert scheme of n-points in X. Generally $\mathrm{Hilb}^n(X)$ is a highly singular space, however there is a wonderful theorem due to Fogarty [37]:

Proposition 4.19. *Let X be a smooth quasi-projective surface, then $Hilb^n(X)$ is a smooth variety of dimension $2n$.*

We denote by $S^n(X)$ the n^{th} symmetric power of X. The Hilbert scheme $\mathrm{Hilb}^n(X)$ parameterizes ideal sheaves $\mathcal{J} \subset \mathcal{O}_X$ such that the support of $\mathcal{O}_x/\mathcal{J}$ in X is a finite collection of points and $\dim \Gamma(X, \mathcal{O}_x/\mathcal{J}) = n$. See [76] for details. There exists a surjective map $\pi : \mathrm{Hilb}^n(X) \twoheadrightarrow S^n(X)$ sending the ideal $\mathcal{J}$ to the support of the $\mathcal{O}_X$-module $\mathcal{O}_X/\mathcal{J}$. This map is a proper morphism and is called the *Hilbert-Chow morphism.* When $X = \mathbb{C}^2$ the Hilbert-Chow morphism $\pi : \mathrm{Hilb}^n(\mathbb{C}^2) \twoheadrightarrow S^n(\mathbb{C}^2) \simeq \mathbb{C}^{2n}/S_n$ is a symplectic resolution.

The two-dimensional varieties $\mathbb{C}^2/\Gamma$, $\Gamma < SL_2(\mathbb{C})$ are the famous Kleinian, or Du-Val, singularities. They have been, and continue to be, extensively studied, especially with regards to the "McKay Correspondence" - see [46], [89], [84], [60]. Since $\dim \mathbb{C}^2/\Gamma = 2$, there exists a unique minimal resolution $\tau : \widetilde{\mathbb{C}^2/\Gamma} \twoheadrightarrow \mathbb{C}^2/\Gamma$ through which all other resolutions factor. This is a symplectic resolution and can either be explicitly constructed through a series of blowups or can be described as the moduli space $\mathrm{Hilb}^{\Gamma}(\mathbb{C}^2)$, [61, Theorem 9.3]. By combining the Hilbert scheme and the minimal resolution, Wang constructs a symplectic resolution for $\mathbb{C}^{2n}/(\Gamma \wr S_n)$.

Figure 4.1: The symplectic resolution of the Kleinian singularity $\mathbb{C}^2/\Gamma$ where Γ is the binary dihedral group of order 8 corresponding to D_4 (the left hand picture is actually a deformation of the singularity).



Proposition 4.20 (Proposition 1, [100]). *Let Γ be a finite subgroup of $SL_2(\mathbb{C})$. The morphism*

$$\tau_n \,:\, Hilb^n(\widetilde{\mathbb{C}^2/\Gamma}) \xrightarrow{\pi_{(n)}} S^n(\widetilde{\mathbb{C}^2/\Gamma}) \xrightarrow{\tau_{(n)}} S^n(\mathbb{C}^2/\Gamma) \simeq \mathbb{C}^{2n}/\Gamma \wr S_n$$

is a symplectic resolution.

Proof. The surface $\widetilde{\mathbb{C}^2/\Gamma}$ is smooth by definition and so Fogarty's Theorem implies that $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$ is smooth. The construction of symmetric powers is functorial so the surjectivity of τ implies that $\tau_{(n)}$ is surjective. Since the Hilbert-Chow morphism is also surjective, τ_n is a resolution of singularities. To show that τ_n is a symplectic resolution [39, Proposition 1.6] says that it suffices to show that $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$ is a symplectic manifold. Beauville [2] has shown that if X is a symplectic surface then $\mathrm{Hilb}^n(X)$ has a symplectic 2-form induced from the 2-form on X. $\square$

After publication of the author's paper [7], Lehn and Sorger [69] constructed two explicit resolutions of the symplectic variety $\mathfrak{h} \times \mathfrak{h}^*/G_4$ using the computer program SINGULAR [54]. We will describe the outcome of their calculations. There are four reflection hyperplanes in each of $\mathfrak{h}$ and $\mathfrak{h}^*$ and we denote by $A_1 \subset \mathfrak{h}$ and $A_2 \subset \mathfrak{h}^*$ the union of these four hyperplanes in each of the spaces $\mathfrak{h}$ and $\mathfrak{h}^*$. Let W_1 denote the image of $A_1 \times \mathfrak{h}^*$ in $\mathfrak{h} \times \mathfrak{h}^*/G_4$ and W_2 the image of $\mathfrak{h} \times A_2$ in $\mathfrak{h} \times \mathfrak{h}^*/G_4$. The subvarieties W_1 and W_2 are Weil divisors in $\mathfrak{h} \times \mathfrak{h}^*/G_4$. For $i = 1, 2$, let $\rho_i \,:\, Z_i' \twoheadrightarrow \mathfrak{h} \times \mathfrak{h}^*/G_4$ denote the blow-up of $\mathfrak{h} \times \mathfrak{h}^*/G_4$ along W_i. Set $W_i' := ((Z_i')_{\mathrm{red}})_{\mathrm{sing}}$ and denote by $\phi_i \,:\, Z_i \to Z_i'$ the blow-up of Z_i' along W_i'.

Proposition 4.21 (Theorem 2, [69]). *The morphisms $\sigma_i := \phi_i \circ \rho_i \,:\, Z_i \twoheadrightarrow \mathfrak{h} \times \mathfrak{h}^*/G_4$ are symplectic resolutions.*

The G_4-Hilbert scheme is the fine moduli space parameterizing the set

$$G_4 - \mathrm{Hilb}(\mathfrak{h} \times \mathfrak{h}^*) := \{\, I \in \mathrm{Hilb}^{24}(\mathfrak{h} \times \mathfrak{h}^*) \,|\, \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]/I \simeq \mathbb{C}G_4 \text{ as a } G_4\text{-module} \,\}.$$

There is a unique irreducible component of the G_4-Hilbert scheme that maps birationally onto $\mathfrak{h} \times \mathfrak{h}^*/G_4$. We denote this component as $\mathrm{Hilb}^{G_4}(\mathfrak{h} \times \mathfrak{h}^*)$. Lehn and Sorger show that the G_4-Hilbert scheme has only one other component, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This component meets $\mathrm{Hilb}^{G_4}(\mathfrak{h} \times \mathfrak{h}^*)$ transversely.

It is shown that the variety $\mathrm{Hilb}^{G_4}(\mathfrak{h} \times \mathfrak{h}^*)$ is smooth and hence gives a resolution of $\mathfrak{h} \times \mathfrak{h}^*/G_4$. However the resolution is not semi-small therefore it cannot be a symplectic resolution. The various resolutions are shown to fit into the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Hilb}^{G_4}(\mathfrak{h} \times \mathfrak{h}^*) \arrow[dl] \arrow[dr] & \\
Z_1 \arrow[dr, "\sigma_1"'] & & Z_2 \arrow[dl, "\sigma_2"] \\
 & \mathfrak{h} \times \mathfrak{h}^*/G_4 &
\end{tikzcd}$$

4.6 Remarks

1. It was shown, before Verbitsky published his result [99], by Kaledin [64] that if $G \subset GL(\mathfrak{h})$ is a finite group such that the symplectic quotient $\mathfrak{h} \times \mathfrak{h}^*/G$ has a symplectic resolution, projective over its base, then G is necessarily a complex reflection group.

2. The image in example 4.5 was generated using the computer package SURFEX [57].

3. The following interesting question was raised in [50, 9. Problems]:
   Q: Give a representation theoretic construction of a symplectic resolution for $\mathfrak{h} \times \mathfrak{h}^*/G_4$*. Does it have a hyper-Hähler structure?*

4. For an overview of the main results and properties of symplectic singularities see the excellent survey [39].

Chapter 5

Cuspidal representations and the Etingof-Ginzburg sheaf

In this chapter we look at a way of relating the representation theory and geometry of a rational Cherednik algebra associated to a group W to the representation theory and geometry of a rational Cherednik algebra associated to a parabolic subgroup of W. The key result that makes this analysis possible is a recent construction of Bezrukavnikov and Etingof [10]. They show that a certain completion of the rational Cherednik algebra is isomorphic to the ring of matrices over a completion of a rational Cherednik algebra associated to a parabolic subgroup of W. The precise statement is given in Theorem 5.14 below. Since this is an isomorphism of complete algebras, in the first part of this chapter we will look at the properties of completions of Poisson algebras and see how these properties compare with the properties of the original uncompleted algebra. Before doing that we present a summary of the main results of the chapter.

5.1 Main results

Let us recall (2.2) the definition of the Etingof-Ginzburg sheaf:

Definition 5.1. The *Etingof-Ginzburg sheaf* is the coherent sheaf $\mathcal{R}[W]$ on $X_{\mathbf{c}}(W)$ corresponding to the finitely generated $Z_{0,\mathbf{c}}(W)$-module $H_{0,\mathbf{c}}(W) \cdot \mathbf{e}$.

Theorem 2.4 says that we can recover the rational Cherednik algebra $H_{0,\mathbf{c}}(W)$ from $\mathcal{R}[W]$,

$$H_{0,\mathbf{c}}(W) = \mathrm{End}_{X_{\mathbf{c}}(W)}(\mathcal{R}[W]).$$

Recall (2.4) that Etingof and Ginzburg constructed an isomorphism between the generalized Calogero-Moser space $X_1(S_n)$ and the classical Calogero-Moser space $\mathcal{C}_n$, as studied by Wilson [102]. In (2.4) we described a map $\pi := \pi_1 : X_{\mathbf{c}}(W) \twoheadrightarrow \mathfrak{h}/W$. Wilson [102, Lemma 7.1] showed:

Lemma 5.2. *Let* $b = b_1 n_1 + \cdots + b_k n_k \in S^n(\mathbb{C}) = \mathbb{C}^n/S_n$*, where* $b_i \in \mathbb{C}$ *are pairwise distinct and* $n_i \in \mathbb{N}$

such that $\sum_{i=1}^{k} n_i = n$. Then there exists an isomorphism of (reduced) varieties

$$\pi_{S_n}^{-1}(b) \simeq \pi_{S_{n_1}}^{-1}(0) \times \cdots \times \pi_{S_{n_k}}^{-1}(0).$$

Based upon this "factorization" result, Etingof and Ginzburg [36, page 319] conjectured that there should be a corresponding factorization of the Etingof-Ginzburg sheaf when restricted to $\pi_{S_n}^{-1}(b)$. The first main result of this chapter says that Wilson's Lemma generalizes to a (scheme theoretic) isomorphism for any group W:

$$\Phi : \pi_W^{-1}(b) \xrightarrow{\sim} \pi_{W_b}^{-1}(0), \tag{5.1}$$

where W_b is a parabolic subgroup of W associated to the orbit $b \in \mathfrak{h}/W$. Then it is shown that Etingof and Ginzburg's conjecture holds in this more general situation. We describe the pushforward of $\mathcal{R}[W]|_{\pi_W^{-1}(b)}$ by Φ.

Theorem 5.3. *On $\pi_{W_b}^{-1}(0)$ there is an isomorphism of W-equivariant sheaves*

$$\Phi_* \left(\mathcal{R}[W]|_{\pi_W^{-1}(b)} \right) \simeq \mathit{Ind}_{W_b}^{W} \mathcal{R}[W_b]|_{\pi_{W_b}^{-1}(0)}$$

A second application of Bezrukavnikov and Etingof's result is to do with those finite dimensional quotients of $H_{\mathbf{c}}(W)$ that are supported on a closed point of $X_{\mathbf{c}}(W)$. Let $\chi \in X_{\mathbf{c}}(W)$ and let $H_{\mathbf{c},\chi} := H_{0,\mathbf{c}}/\mathfrak{m}_\chi \cdot H_{0,\mathbf{c}}$ be the "largest" quotient of $H_{0,\mathbf{c}}$ supported at χ (here $\mathfrak{m}_\chi$ is the maximal ideal of $Z_{\mathbf{c}}(W)$ defining χ). Let $\mathcal{L}$ denote the symplectic leaf on which χ sits. If $\mathcal{L}$ is a zero-dimensional leaf, $\mathcal{L} = \{\chi\}$, then we call $H_{\mathbf{c},\chi}$ a *cuspidal algebra*. Our main result in this direction is:

Theorem 5.4. *Let $\mathcal{L}$ be a leaf in $X_{\mathbf{c}}(W)$ of dimension $2l$ and χ a point on $\mathcal{L}$. Then there exists a parabolic subgroup W_b, $b \in \mathfrak{h}$, of W of rank $\dim \mathfrak{h} - l$ and a cuspidal algebra $H_{\mathbf{c}',\psi}$ with $\psi \in X_{\mathbf{c}'}(W_b)$ such that*

$$H_{\mathbf{c},\chi} \simeq \mathit{Mat}_{|W/W_b|}\left(H_{\mathbf{c}',\psi}\right).$$

As a consequence we show that:

Corollary 5.5. *Let $\chi, \mathcal{L}, W_b$ and ψ be as in the above theorem. Then there exists a functor*

$$\Phi_{\psi,\chi} \, : \, H_{\mathbf{c}',\psi}\text{-mod} \xrightarrow{\sim} H_{\mathbf{c},\chi}\text{-mod}$$

defining an equivalence of categories such that

$$\Phi_{\psi,\chi}(M) \simeq \mathit{Ind}_{W_b}^{W} M \quad \forall\, M \in H_{\mathbf{c}',\psi}\text{-mod}$$

as W-modules.

This shows that the problem of describing the W-module structure of the simple $H_{0,\mathbf{c}}(W)$-modules reduces to studying the simple modules of the cuspidal algebras. We show that these simple modules always occur as a simple module for the restricted rational Cherednik algebra.

5.2 Poisson Ideals

In this section we state and prove certain results on completed Poisson algebras that will be required later. Throughout R will denote a commutative, affine domain over a field k. If I is a proper ideal of R then Krull's Intersection Theorem ([34, Corollary 5.4]) says that

$$\bigcap_{n=1}^{\infty} I^n = 0.$$

Therefore, if $\widehat{R}_I$ denotes the completion of R along I, the natural map $j : R \to \widehat{R}_I$ is an inclusion. The Krull dimension of R will be written $\text{Kl.dim}\, R$.

Lemma 5.6. *For R, I as above,*

$$\text{Kl.dim}\, R = \text{Kl.dim}\, \widehat{R}_I$$

Proof. Let $\mathfrak{n}$ be a maximal ideal of $\widehat{R}_I$, then [53, Corollary 2.19] shows that $\mathfrak{n} \mapsto \mathfrak{n} \cap R$ defines a bijection between the maximal ideals of $\widehat{R}_I$ and the maximal ideals of R containing I. Moreover, the proof of [53, Theorem 7.5] says that $\text{ht}\,(\mathfrak{n}) = \text{ht}\,(\mathfrak{n} \cap R)$. Therefore $\text{Kl.dim}\, \widehat{R}_I = \sup\{\text{ht}\,(\mathfrak{m})\}$, where $\mathfrak{m}$ ranges over all maximal ideals of R that contain I. Since R is an affine domain over k, [34, Theorem A]) says that $\text{ht}\,(\mathfrak{m}) = \text{Kl.dim}\, R$ for all maximal ideals of R, hence $\text{Kl.dim}\, R = \text{Kl.dim}\, \widehat{R}_I$. □

It will be particularly important for us later to understand what happens to prime ideals when passing to completions.

Lemma 5.7. *Choose a prime ideal $P \lhd R$ such that $P \otimes_R \widehat{R}_I \neq \widehat{R}_I$ and Q a prime ideal of $\widehat{R}_I$. Then*

1. *For each prime Q' minimal over $P \otimes_R \widehat{R}_I$, $ht(Q') = ht(P)$ and $Q' \cap R = P$.*

2. *$Q \cap R$ is a prime ideal and $ht(Q) = ht(Q \cap R)$.*

3. *If $I \subseteq P$ then $P \otimes_R \widehat{R}_I$ is prime in $\widehat{R}_I$.*

Proof. Clearly $Q \cap R$ is a prime ideal. By [34, Theorem 7.2], $\widehat{R}_I$ is a flat extension of R therefore [34, Lemma 10.11] shows that (Going down) holds. Now let Q' be a prime minimal over $P \otimes_R \widehat{R}_I$. If $Q' \cap R \neq P$ then by (Going down) there exists a prime $\tilde{Q} \subsetneq Q'$ such that $\tilde{Q} \cap R = P \subsetneq Q' \cap R$. But then $P \otimes_R \widehat{R}_I \subset \tilde{Q}$, contradicting the minimality of Q'. Fix a maximal chain of primes $P_0 \supset P_1 \supset \cdots \supset P_n = 0$ such that $P_j = Q \cap R$ for some fixed j and $I \subseteq P_0$. By [34, Theorem A, page 286], R is universally caternary, hence $n = \text{Kl.dim}\, R$. The result [53, Corollary 2.19] says that there is a unique maximal ideal Q_0 of $\widehat{R}_I$ such that $Q_0 \cap R = P_0$. The proof of Lemma 5.6 shows that $\text{Kl.dim}\, R = \text{ht}\,(P_0) = \text{ht}\,(Q_0) = \text{Kl.dim}\, \widehat{R}_I$. Applying (Going down) to $P_0 \supset P_1$ shows that there exists a prime Q_1 such that $Q_1 \cap R = P_1$ and $Q_1 \subsetneq Q_0$. Clearly $\text{ht}\,(P_1) \geq \text{ht}\,(Q_1)$. By repeating this argument we get a chain of primes $Q_0 \supset Q_1 \supset \cdots \supset Q_n$ such that $Q_i \cap R = P_i$ and $\text{ht}\,(P_i) \geq \text{ht}\,(Q_i)$. But Lemma 5.6 implies that we must have $\text{ht}\,(Q_i) = \text{ht}\,(P_i)$. In particular, $\text{ht}\,(Q) = \text{ht}\,(Q \cap R)$. This completes the proof of (1) and (2).
By [34, Theorem 7.2], $\widehat{P} := P \otimes_R \widehat{R}_I = \lim_{\infty \leftarrow n} P/I^n$ (note that $I \subset P$ implies $\widehat{P} \neq \widehat{R}_I$). Let us show that $\widehat{P}$ is prime. If not then there exist $a, b \in \widehat{R}_I \backslash \widehat{P}$ such that $a \cdot b \in \widehat{P}$. Therefore there exists some

$N > 0$ such that $\bar{a}, \bar{b} \in (R/I^N) \setminus (P/I^N)$ with $\bar{a} \cdot \bar{b} \in P/I^N$. But this is a contradiction since P/I^N is prime. $\square$

If S_1 and S_2 are k-algebras, complete with respect to the ideals I_1 and I_2 respectively then the completed tensor product is defined to be

$$S_1 \widehat{\otimes} S_2 := \lim_{\infty \leftarrow n} (S_1 \otimes S_2)/J^n,$$

where $J := I_1 \otimes S_2 + S_1 \otimes I_2$.

Lemma 5.8. *Let P be a prime ideal of $\widehat{R}_I$ and Q the ideal generated by P in $\widehat{R}_I \widehat{\otimes} k[[x]]$. Then Q is prime.*

Proof. Since $\widehat{R}_I$ is Noetherian, the ideal P is finitely generated. By [34, Theorem 7.2],

$$Q = \lim_{\infty \leftarrow n} P \otimes k[x]/J^n = (P \otimes k[x]) \otimes_{\widehat{R}_I \otimes k[x]} \widehat{R}_I \widehat{\otimes} k[[x]] = \{ \sum_{i \geq 0} p_i x^i \mid p_i \in P \},$$

is a finitely generated ideal in $\widehat{R}_I \widehat{\otimes} k[[x]]$, where $J = I \otimes k[x] + R \otimes (x)$. Now choose $a = \sum_{i \geq 0} a_i x^i, b = \sum_{j \geq 0} b_j x^j \in \widehat{R}_I \widehat{\otimes} k[[x]]$ such that $a \cdot b \in Q$. If $a, b \notin Q$ then we can choose $r, s \in \mathbb{N}$ to be minimal with respect to the properties $a_r, b_s \notin P$. Then the fact that the coefficient of x^{r+s} in the expansion of $a \cdot b$ lies in P is a contradiction. $\square$

For the reminder of this section we make the additional assumptions that R is a Poisson algebra with bracket $\{\cdot, \cdot\}$ and that $k = \mathbb{C}$.

Lemma 5.9. *Let R, I be as above. We do not assume that I is a Poisson ideal.*

1. *$\widehat{R}_I$ is a Poisson algebra.*
2. *If Q is a Poisson prime of $\widehat{R}_I$ then $Q \cap R$ is a Poisson prime.*
3. *If J is a Poisson ideal such that $J \otimes_R \widehat{R}_I \neq \widehat{R}_I$ then $J \otimes_R \widehat{R}_I$ is a Poisson ideal and any prime minimal over $J \otimes_R \widehat{R}_I$ is Poisson.*

Proof. Each element of $\widehat{R}_I$ has the form $(f_i)_{i \in \mathbb{N}}$, where $f_i \in R/I^i$ and $f_j \equiv f_i \bmod I^i$ for all $j > i$. The Poisson structure on $\widehat{R}_I$, (denoted $\langle \cdot, \cdot \rangle$) is defined as $\langle f, g \rangle_i := \{f_{i+1}, g_{i+1}\} + I^i$ (alternatively one can simply note that, for fixed $f \in R$, $\{f, -\}$ is a derivation of R and thus continuous in the I-adic topology). Denote by $\iota : R \hookrightarrow \widehat{R}_I$ the inclusion map. Let $f, g \in R$, then $\langle \iota(f), \iota(g) \rangle_i = \langle f + I^{i+1}, g + I^{i+1} \rangle_i = \{f, g\} + I^i$. Therefore $\langle \iota(f), \iota(g) \rangle = \iota(\{f, g\})$ and (2) follows from this.
To show that $J \otimes_R \widehat{R}_I$ is a Poisson ideal, choose $(f_i)_{i \in \mathbb{N}} \in J \otimes_R \widehat{R}_I$ and $(g_i)_{i \in \mathbb{N}} \in \widehat{R}_I$. Then, for each i in $\mathbb{N}$, there exists $p_i \in J$ such that $p_i \equiv f_i \bmod I^i$ and $\langle (f_i), (g_i) \rangle_i = \{f_{i+1}, g_{i+1}\} + I^i = \{p_{i+1}, g_{i+1}\} + I^i \in (J + I^i)/I^i$. Hence $\langle J \otimes_R \widehat{R}_I, \widehat{R}_I \rangle \subset J \otimes_R \widehat{R}_I$. Noting that $k = \mathbb{C}$, [31, Lemma 3.3.3] says that the primes minimal over $J \otimes_R \widehat{R}_I$ are Poisson. $\square$

We can compare the Poisson cores in R with those in $\widehat{R}_I$ as follows:

Lemma 5.10. *Let R and I be as above and choose $\mathfrak{m}$ a maximal ideal of R containing I. Then every prime minimal over $\mathcal{C}(\mathfrak{m}) \otimes_R \widehat{R}_I$ is Poisson primitive and the Poisson core of $\mathfrak{m} \otimes_R \widehat{R}_I$ is one of these minimal primes. Conversely, if J is a Poisson primitive ideal in $\widehat{R}_I$ then $J \cap R$ is Poisson primitive.*

Proof. By [53, Corollary 2.19], $I \subset \mathfrak{m}$ implies that $\widehat{R}_I \neq \mathfrak{m} \otimes_R \widehat{R}_I$ is a maximal ideal of $\widehat{R}_I$. Therefore $\mathcal{C}(\mathfrak{m}) \otimes_R \widehat{R}_I$ is also a proper ideal of $\widehat{R}_I$, which is Poisson by Lemma 5.9. Let P be a prime minimal over $\mathcal{C}(\mathfrak{m}) \otimes_R \widehat{R}_I$. Again by Lemma 5.9, it is Poisson. Since [53, Corollary 2.19] says that there is a bijection between maximal ideals of $\widehat{R}_I$ and maximal ideals of R containing I it suffices to consider the case $P \subseteq \mathfrak{m} \otimes_R \widehat{R}_I$. If $\mathcal{C}(\mathfrak{m}) = \mathfrak{m}$ then the result is trivial. Therefore, without loss of generality, $\mathcal{C}(\mathfrak{m}) \subsetneq \mathfrak{m}$. Assume that P is not the Poisson core of $\mathfrak{m} \otimes_R \widehat{R}_I$, so that $P \subsetneq Q = \mathcal{C}(\mathfrak{m} \otimes_R \widehat{R}_I) \subseteq \mathfrak{m} \otimes_R \widehat{R}_I$. By Lemma 5.7, $\mathcal{C}(\mathfrak{m}) = R \cap P \subseteq Q \cap R \subseteq \mathfrak{m} \otimes_R \widehat{R}_I \cap R = \mathfrak{m}$, and Lemma 5.9 says that $Q \cap R$ is a Poisson prime. Therefore $Q \cap R = \mathcal{C}(\mathfrak{m})$ by maximality. But Lemma 5.7 says that

$$\operatorname{ht} \mathcal{C}(\mathfrak{m}) = \operatorname{ht}(P) < \operatorname{ht}(Q) = \operatorname{ht}(Q \cap R).$$

This contradiction shows that P is Poisson primitive. The same argument also implies the converse statement. □

5.3 The setup

The aim of the next section will be to prove that Bezrukavnikov and Etingof's isomorphism induces an isomorphism between a certain completion of the centre of $H_{0,\mathbf{c}}(W)$ and a completion of the centre of the rational Cherednik algebra associated to a parabolic subgroup of W. To make the exposition clearer we prove the required result in a slightly more abstract setup. For $i = 1, 2$ we choose A_i to be a $\mathbb{C}$-algebra, $\mathbf{t}_i \in \mathsf{A}_i$ a central non-zero divisor and $\rho_i : \mathsf{A}_i \twoheadrightarrow A_i := \mathsf{A}_i/\mathbf{t}_i\mathsf{A}_i$. Assume that there exists a finite dimensional, abelian Lie subalgebra $\mathfrak{n}_i$ of A_i such that the adjoint action of $\mathfrak{n}_i$ on A_i is locally nilpotent. Denote by $\mathcal{U}_{i,+}$ the associative subalgebra (without unit) in A_i generated by $\mathfrak{n}_i$ and let $\mathcal{U}_{i,+}^k$ be the k^{th} power of $\mathcal{U}_{i,+}$ ($k \in \mathbb{N}$). As noted in [43, (5.1)], for any $a \in \mathsf{A}_i$ there exists $n \in \mathbb{Z}$ (depending on a) such that

$$a \cdot \mathcal{U}_{i,+}^k \subset \mathcal{U}_{i,+}^{k+n} \cdot \mathsf{A}_i \qquad \forall k \gg 0. \tag{5.2}$$

We make the additional assumption that the image of $\mathfrak{n}_i$ under ρ_i is contained in the centre Z_i of A_i. The ideal generated in Z_i by $\rho_i(\mathfrak{n}_i)$ will be denoted I_i. We assume that Z_i is affine and A_i a finite module over Z_i. Property (5.2) implies that the space

$$\widehat{\mathsf{A}}_i := \lim_{\infty \leftarrow k} \mathsf{A}_i \,/\, \mathcal{U}_{i,+}^k \cdot \mathsf{A}_i, \quad i = 1, 2$$

is an associative algebra that is complete with respect to the topology on A_i defined by the set $\{\mathcal{U}_{i,+}^k \cdot \mathsf{A}_i\}_{k \geq 1}$ of fundamental neighborhoods of zero.

Finally, we assume that there exists an isomorphism

$$\theta\,:\,\widehat{\mathsf{A}}_1 \xrightarrow{\sim} \widehat{\mathsf{A}}_2$$

such that $\theta(\mathbf{t}_1) = \mathbf{t}_2$ and $\theta(\mathcal{U}_{1,+}^k \cdot \widehat{\mathsf{A}}_1) = \mathcal{U}_{2,+}^k \cdot \widehat{\mathsf{A}}_2$ for all $k \geq 0$ (thus θ is a homeomorphism). We write $\widehat{A}_i := \widehat{\mathsf{A}}_i \,/\, \mathbf{t}_i \cdot \widehat{\mathsf{A}}_i$ and let $\widehat{Z}_i$ be the completion of Z_i with respect to the ideal I_i.

Lemma 5.11. *Let A_i, $\mathcal{U}_{i,+}$, Z_i and I_i be as above. Then*

$$Z(\widehat{A}_i) = \widehat{Z}_i.$$

Proof. Since Z_i is a Noetherian ring, $\widehat{Z}_i$ is a flat Z_i-module and $\widehat{A}_i = A_i \otimes_{Z_i} \widehat{Z}_i$. We choose a generating set $a_1, \ldots, a_n$ of A_i as a module over Z_i and assume without loss of generality that $a_1 = 1$. The flatness of $\widehat{Z}_i$ implies that the natural map $\widehat{Z}_i \to \widehat{A}_i$ is an embedding. Its image is central, therefore it suffices to show that $Z(\widehat{A}_i) \subseteq \widehat{Z}_i$. Let h be central in $\widehat{A}_i$. We prove by induction on $1 \leq l \leq n$ that there exist $h_j \in A$ and $z_j \in \widehat{Z}_i$ such that $h = \sum_j h_j \otimes z_j$ and the h_j's commute with every a_t, $t \leq l$. This is clear when $l = 1$. Therefore assume $l > 1$ and that there exist h_j, z_j such that $h = \sum_j h_j \otimes z_j$ and the h_j's commute with all a_t, $t < l$. Since $\sum_j [a_l, h_j] \otimes z_j = 0$, the flatness of $\widehat{Z}_i$ implies that there exist $b_{jk} \in Z_i$ and $z'_k \in \widehat{Z}_i$ such that

1. $\sum_k b_{jk} z'_k = z_j$ in $\widehat{Z}_i$,
2. $\sum_j [a_l, h_j] b_{jk} = 0$ in A_i i.e. $[a_l, \sum_j h_j b_{jk}] = 0$.

Therefore $h'_k := \sum_j h_j b_{jk}$ commutes with $a_1, \ldots, a_{l-1}, a_l$. However (1) also implies that $h = \sum_k h'_k \otimes z'_k$. Induction implies that $h \in \widehat{Z}_i$. $\square$

Proposition 5.12. *Assume that Z_i is a direct summand of A_i as a Z_i-module. The isomorphism θ induces a* ***Poisson*** *isomorphism*

$$\theta\,:\,\widehat{Z}_1 \xrightarrow{\sim} \widehat{Z}_2$$

Proof. Since $\theta(\mathbf{t}_1) = \mathbf{t}_2$, θ defines an isomorphism $\widehat{A}_1 \xrightarrow{\sim} \widehat{A}_2$. This restricts to an isomorphism of the centres. By Lemma 5.11, $Z(\widehat{A}_i) = \widehat{Z}_i$, and θ induces an isomorphism $\widehat{Z}_1 \xrightarrow{\sim} \widehat{Z}_2$. Therefore we must show that θ is a Poisson morphism. Let $u, v \in \widehat{Z}_1$, $u = (u_i)_{i \geq 0}$ and $v = (v_i)_{i \geq 0}$ where $u_i, v_i \in Z_1 \,/\, I_1^i$ and choose lifts of u, v to $\hat{u}$ and $\hat{v}$ in $\widehat{\mathsf{A}}_1$. The fact that θ induces an isomorphism $\widehat{Z}_1 \cong \widehat{Z}_2$ together with the fact that $\theta \circ \rho_1 = \rho_2 \circ \theta$ (since $\theta(\mathbf{t}_1) = \mathbf{t}_2$) imply that $\theta(\hat{u})$ is a lift of $\theta(u)$. The assumption that Z_i is a direct summand of A_i as a Z_i-module implies that $Z_i \cap (\mathcal{U}_{i,+}^k \cdot A_i) = I_i^k$, hence

$$Z_i \,/\, I_i^k \hookrightarrow A_i \,/\, \mathcal{U}_{i,+}^k \cdot A_i \quad \forall\, k \geq 0.$$

We recall the definition of the Poisson bracket on $\widehat{Z}_i$ (combining Lemma 5.9 and equation (1.1)):

$$(\{u, v\})_i := \rho_1([\hat{u}_{i+1}, \hat{v}_{i+1}]/\mathbf{t}_1) \bmod I_1^i.$$

Now

$$\begin{aligned}
(\theta(\{u,v\}))_i &= \theta(\rho_1([\hat{u}_{i+1},\hat{v}_{i+1}]/\mathbf{t}_1) \bmod I_1^i) \\
&= \theta(\rho_1([\hat{u}_{i+1},\hat{v}_{i+1}]/\mathbf{t}_1) \bmod \mathcal{U}_{1,+}^i \cdot A_1) \\
&= \theta(\rho_1([\hat{u}_{i+1},\hat{v}_{i+1}]/\mathbf{t}_1)) \bmod \mathcal{U}_{2,+}^i \cdot A_2 \\
&= \rho_2(\theta([\hat{u}_{i+1},\hat{v}_{i+1}])/\mathbf{t}_2)) \bmod \mathcal{U}_{2,+}^i \cdot A_2 \\
&= \rho_2([\theta(\hat{u}_{i+1}),\theta(\hat{v}_{i+1})]/\mathbf{t}_2)) \bmod \mathcal{U}_{2,+}^i \cdot A_2 \\
&= \rho_2([\theta(\hat{u}_{i+1}),\theta(\hat{v}_{i+1})]/\mathbf{t}_2)) \bmod I_2^i \\
&= (\{\theta(u),\theta(v)\})_i,
\end{aligned}$$

where in the second and sixth line we have used the fact that $Z_i \,/\, I_i^k \hookrightarrow A_i \,/\, \mathcal{U}_{i,+}^k \cdot A_i$, in the fourth line we use the fact that $\theta \circ \rho_1 = \rho_2 \circ \theta$ and in the final line we use the fact that $\theta(\hat{u})$ is a lift of $\theta(u)$ to A_2. $\square$

5.4 Completions of the generalized Calogero-Moser Space

In the remainder of this chapter it will be necessary to consider rational Cherednik algebras associated to the same complex reflection group but with different reflection representations. Therefore to avoid any ambiguity we will write $H_{\mathbf{c}}(W,\mathfrak{h})$, $Z_{\mathbf{c}}(W,\mathfrak{h})$, $X_{\mathbf{c}}(W,\mathfrak{h})$ and so on, to keep track of this additional information. Let $U \subset \mathbb{C}[\mathfrak{h}]^W$ be a minimal, homogeneous generating subspace of $\mathbb{C}[\mathfrak{h}]^W$. Let $b \in \mathfrak{h}$ and $\lambda \in U$, then we can evaluate λ on the orbit $W \cdot b$, $b \mapsto \lambda(b)$. Let $\mathfrak{m}(b) := \{\lambda - \lambda(b) \,|\, \lambda \in U\}$. The ideal generated by $\mathfrak{m}(b)$ in $\mathbb{C}[\mathfrak{h}]^W$ is the maximal ideal corresponding to the orbit $W \cdot b \in \mathfrak{h}/W$. Similarly, if W_b is the stabilizer of b in W, choose $U_b \subset \mathbb{C}[\mathfrak{h}]^{W_b}$ to be a minimal, homogeneous generating subspace of $\mathbb{C}[\mathfrak{h}]^{W_b}$ and let $\mathfrak{n}(q) := \{\lambda - \lambda(q) \,|\, \lambda \in U_b\}$ for each $q \in \mathfrak{h}$. As noted in [43, Section 6], we are in the setup of (5.3) if we take $\mathsf{A}_1 = H_{\mathbf{t},\mathbf{c}}(W,\mathfrak{h})$, $\mathfrak{n}_1 = \mathfrak{m}(b)$, $\mathsf{A}_2' = H_{\mathbf{t},\mathbf{c}'}(W_b,\mathfrak{h})$ and $\mathfrak{n}_2' = \mathfrak{n}(0)$. Thus we get complete, associative algebras

$$\widehat{H}_{\mathbf{t},\mathbf{c}}(W,\mathfrak{h})_b := \lim_{\infty \leftarrow k} H_{\mathbf{t},\mathbf{c}}(W,\mathfrak{h})/\mathfrak{m}(b)^k \cdot H_{\mathbf{t},\mathbf{c}}(W,\mathfrak{h}),$$

$$\widehat{H}_{\mathbf{t},\mathbf{c}'}(W_b,\mathfrak{h})_0 := \lim_{\infty \leftarrow k} H_{\mathbf{t},\mathbf{c}'}(W_b,\mathfrak{h})/\mathfrak{n}(0)^k \cdot H_{\mathbf{t},\mathbf{c}'}(W_b,\mathfrak{h}).$$

To get A_2, $\mathfrak{n}_2$ and θ we need to introduce a certain centralizer algebra.

5.5 Centralizer algebras

We recall the centralizer construction described in [10, 3.2]. Let A be an $\mathbb{C}$-algebra equipped with an injective homomorphism $H \longrightarrow A^{\times}$, where H is a finite group. Let G be another finite group such that H is a subgroup of G. The algebra $C(G,H,A)$ is defined to be the centralizer of A in the right A-module $P := \mathrm{Fun}_H(G,A)$ of H-invariant, A-valued functions on G. By making a choice of left coset representatives of H in G, $C(G,H,A)$ is realized as the algebra of $|G/H|$ by $|G/H|$ matrices over A. For $w,g \in G$ and $f \in \mathrm{Fun}_H(G,A)$, $w \cdot f(g) := f(gw)$ defines, by linearity, an embedding $\iota : \mathbb{C}G \hookrightarrow C(G,H,A)$. Let $\mathbf{e}_G \in \mathbb{C}G$ and $\mathbf{e}_H \in \mathbb{C}H$ denote the idempotents corresponding to the trivial representation of G and H respectively, where $\mathbb{C}H$ is considered as a subalgebra of A.

Lemma 5.13. *There are isomorphisms of $\mathbb{C}G$-$Z(A)$-bimodules*

$$C(G,H,A)\cdot\iota(\mathbf{e}_G)\simeq \mathit{Fun}_H(G,A\mathbf{e}_H)\simeq \mathtt{Ind}_H^G\,A\mathbf{e}_H,$$

where $Z(A)$ denotes the centre of A. Here $\mathbb{C}G$ acts on $C(G,H,A)$ by multiplication on the left via ι and on the left of $\mathit{Fun}_H(G,A\mathbf{e}_H)$ also via ι.

Proof. The second isomorphism is clear from the definition of $\mathrm{Fun}_H(G,A)$. Let $\delta\in\mathrm{Fun}_H(G,A)$ be the function defined by $\delta(g)=\mathbf{e}_H$, for all $g\in G$. We define a linear map ζ from $C(G,H,A)\cdot\iota(\mathbf{e}_G)$ to $\mathrm{Fun}_H(G,A\mathbf{e}_H)$ and a map η in the opposite direction by

$$\zeta:\quad M\cdot\iota(\mathbf{e}_G)\mapsto M(\delta)$$

$$\eta:\quad f\mapsto\left(h(-)\mapsto f(-)\sum_{g\in G}h(g)\right),$$

where $M\in C(G,H,A)$, $f\in\mathrm{Fun}_H(G,A\mathbf{e}_H)$ and $h\in\mathrm{Fun}_H(G,A)$. After fixing left coset representatives of H in G, a direct calculation which we leave to the appendix (A.1) shows that η is both a left and right inverse to ζ. The G-equivariance of ζ is clear since

$$g\cdot\zeta(M\iota(\mathbf{e}_G))=g\cdot M(\delta)=\iota(g)(M(\delta))=(\iota(g)M)(\delta)=\zeta(g\cdot M\iota(\mathbf{e}_G))$$

The $Z(A)$-equivariance of ζ is similarly clear. □

The results of this chapter are all based on the simple observation that [10, Theorem 3.2] is independent of the parameter t and hence can be applied to the case $t=0$. We state it here for completeness.

Theorem 5.14 ([10], Theorem 3.2)**.** *Let $b\in\mathfrak{h}$, and define $\mathbf{c}'$ to be the restriction of $\mathbf{c}$ to the set S_b of reflections in W_b. Then one has an isomorphism of $\mathbb{C}[\mathbf{t}]$-algebras*

$$\theta:\widehat{H}_{\mathbf{t},\mathbf{c}}(W,\mathfrak{h})_b\to C(W,W_b,\widehat{H}_{\mathbf{t},\mathbf{c}'}(W_b,\mathfrak{h})_0),\tag{5.3}$$

defined by the following formulas. Suppose that $f\in\mathit{Fun}_{W_b}(W,\widehat{H}_{\mathbf{t},\mathbf{c}'}(W_b,\mathfrak{h})_0)$. Then

$$(\theta(u)f)(w)=f(wu),u\in W;$$

for any $\alpha\in\mathfrak{h}^*$,

$$(\theta(x_\alpha)f)(w)=(x^{(b)}_{w\alpha}+(w\alpha,b))f(w),$$

where $x_\alpha\in\mathfrak{h}^*\subset H_{\mathbf{t},\mathbf{c}}(W,\mathfrak{h}),x^{(b)}_{w\alpha}\in H_{\mathbf{t},\mathbf{c}'}(W_b,\mathfrak{h})$; and for any $a\in\mathfrak{h}$,

$$(\theta(y_a)f)(w)=y^{(b)}_{wa}f(w)+\sum_{s\in S:s\notin W_b}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(wa)}{x^{(b)}_{\alpha_s}+\alpha_s(b)}(f(sw)-f(w)).$$

where $y_a\in\mathfrak{h}\subset H_{\mathbf{t},\mathbf{c}}(W,\mathfrak{h})$ and $y^{(b)}_a$ the same vector considered now as an element of $H_{\mathbf{t},\mathbf{c}'}(W_b,\mathfrak{h})$.

Let $\mathcal{A}$ denote the set of reflecting hyperplanes of W in $\mathfrak{h}$ and, for each $H \in \mathcal{A}$, let $L_H \in \mathfrak{h}^*$ be a linear functional whose kernel is H (e.g. $\alpha_s \in \mathfrak{h}^*$ if s is a reflection about H). Choose homogeneous algebraically independent generators $F_1, \dots F_n$ of $\mathbb{C}[\mathfrak{h}]^W$ and $P_1, \dots, P_n$ of $\mathbb{C}[\mathfrak{h}]^{W_b}$. The following description of the Jacobian is due to Steinberg, [93, Lemma].

$$\Pi_W := \det\left(\frac{\partial F_i}{\partial x_j}\right) = k \prod_{H \in \mathcal{A}} L_H^{e_H - 1} \tag{5.4}$$

where e_H is the order of the cyclic group W_H of elements of W that fix H point-wise and k a non-zero scalar.

Lemma 5.15. *For each $b \in \mathfrak{h}$ the map $\Psi : \mathbb{C}[[\mathfrak{h}/W_b]]_0 \longrightarrow \mathbb{C}[[\mathfrak{h}/W_b]]_0$ defined by*

$$P_i(\mathbf{x}) \mapsto F_i(\mathbf{x} + b) - F_i(b)$$

is an automorphism.

Proof. Since $F_i(\mathbf{x} + b) - F_i(b) \in \mathfrak{n}(0)$ for all i there exist polynomials $Q_1, \dots, Q_n$ such that $F_i(\mathbf{x} + b) - F_i(b) = Q_i(P_1, \dots, P_n)$. The chain rule gives

$$D := \det\left(\frac{\partial(F_i(\mathbf{x} + b) - F_i(b))}{\partial x_j}\right) = \det\left(\frac{\partial Q_i}{\partial P_k}\right) \det\left(\frac{\partial P_k}{\partial x_j}\right).$$

However, $D = \Pi_W(\mathbf{x} + b)$ and this gives

$$\prod_{H \in \mathcal{A}} L_H^{e_H - 1}(\mathbf{x} + b) = \det\left(\frac{\partial Q_i}{\partial P_k}\right) \prod_{H \in \mathcal{A} \text{ with } b \in H} L_H^{e_H - 1}(\mathbf{x}).$$

Since $L_H(\mathbf{x} + b) = L_H(\mathbf{x})$ if and only if $b \in H$, we get

$$\det\left(\frac{\partial Q_i}{\partial P_k}\right) = \prod_{H \in \mathcal{A} \text{ with } b \notin H} L_H^{e_H - 1}(\mathbf{x} + b)$$

and

$$\det\left(\frac{\partial Q_i}{\partial P_k}\right)(0) = \prod_{H \in \mathcal{A} \text{ with } b \notin H} L_H^{e_H - 1}(b) \neq 0.$$

Hence, by [34, Exercise 7.25], Ψ is an isomorphism. $\square$

Proposition 5.16. *Let $\theta : \widehat{H}_{t,\mathbf{c}}(W, \mathfrak{h})_b \to C(W, W_b, \widehat{H}_{t,\mathbf{c}'}(W_b, \mathfrak{h})_0)$ be the isomorphism (5.3). Then*

$$\theta(\mathfrak{m}(b)^k \cdot H_{t,\mathbf{c}}(W, \mathfrak{h})) = C\left(W, W_b, \mathfrak{n}(0)^k \cdot H_{t,\mathbf{c}'}(W_b, \mathfrak{h})\right), \qquad \forall k \geq 1.$$

Proof. For $a \in \mathfrak{h}$, $\alpha \in \mathfrak{h}^*$ and $w \in W$, $(x_{w \cdot \alpha} + (w\alpha, b))(a) = (w\alpha, a) + (w\alpha, b) = (w \cdot x_\alpha)(a + b)$. Therefore $\theta(g)(f(w)) = (w \cdot g)(\mathbf{x} + b) f(w) = g(\mathbf{x} + b) f(w)$ for all $g \in \mathbb{C}[\mathfrak{h}]^W \subset \mathbb{C}[[\mathfrak{h}]]_b$ and $f \in \mathrm{Fun}_{W_b}(W, \widehat{H}_c(W_b, \mathfrak{h})_0)$. Now choose $u \in W_b$, then

$$u \cdot g(\mathbf{x} + b) = g(u^{-1} \cdot \mathbf{x} + b) = g(u^{-1} \cdot (\mathbf{x} + b)) = g(\mathbf{x} + b)$$

shows that $g(\mathbf{x}+b) \in \mathbb{C}[\mathfrak{h}]^{W_b}$. Hence, if $g \in \mathfrak{m}(b) \lhd \mathbb{C}[\mathfrak{h}]^W$, then $g(\mathbf{x}+b) \in \mathfrak{n}(0) \lhd \mathbb{C}[\mathfrak{h}]^{W_b}$. This shows that $\theta(g)f(w) \in \mathfrak{n}(0)^k \cdot \widehat{H}_{t,\mathbf{c}'}(W_b)_0$ and

$$\theta(\mathfrak{m}(b)^k \cdot \widehat{H}_{t,\mathbf{c}}(W,\mathfrak{h})_b) \subseteq C(W, W_b, \mathfrak{n}(0)^k \cdot \widehat{H}_{t,\mathbf{c}'}(W_b,\mathfrak{h})_0). \tag{5.5}$$

The ideal $\mathfrak{m}(b)$ in $\mathbb{C}[\mathfrak{h}]^W$ is generated by $F_1(\mathbf{x}) - F_1(b), \dots, F_n(\mathbf{x}) - F_n(b)$ and we have $\theta(F_i(\mathbf{x}) - F_i(b))f(w) = (F_i(\mathbf{x}+b) - F_i(b))f(w)$. The statement of Lemma 5.15 is equivalent to the fact that

$$\{F_1(\mathbf{x}+b) - F_1(b), \dots, F_n(\mathbf{x}+b) - F_n(b))\} \cdot \mathbb{C}[[\mathfrak{h}/W_b]]_0 = \mathfrak{n}(0) \cdot \mathbb{C}[[\mathfrak{h}/W_b]]_0,$$

which in turn implies that

$$\{F_1(\mathbf{x}+b) - F_1(b), \dots, F_n(\mathbf{x}+b) - F_n(b))\}^k \cdot \mathbb{C}[[\mathfrak{h}/W_b]]_0 = \mathfrak{n}(0)^k \cdot \mathbb{C}[[\mathfrak{h}/W_b]]_0.$$

This, together with (5.5), implies that

$$\theta(\mathfrak{m}(b)^k \cdot \widehat{H}_{t,\mathbf{c}}(W,\mathfrak{h})_b) = C(W, W_b, \mathfrak{n}(0)^k \cdot \widehat{H}_{t,\mathbf{c}'}(W_b,\mathfrak{h})_0).$$

□

Let us denote by $\widehat{Z}_{\mathbf{c}}(W,\mathfrak{h})_b$ the completion of $Z_{\mathbf{c}}(W,\mathfrak{h})$ with respect to the ideal generated by $\mathfrak{m}(b)$. Similarly, let $\widehat{Z}_{\mathbf{c}'}(W_b,\mathfrak{h})_0$ be the completion of $Z_{\mathbf{c}'}(W_b,\mathfrak{h})$ with respect to the ideal generated by $\mathfrak{n}(0)$. Lemma 5.11 says that

$$Z(\widehat{H}_{0,\mathbf{c}}(W,\mathfrak{h})_b) = \widehat{Z}_{\mathbf{c}}(W,\mathfrak{h})_b \quad \text{and} \quad Z(C(W, W_b, \widehat{H}_{0,\mathbf{c}'}(W_b,\mathfrak{h})_0)) = \widehat{Z}_{\mathbf{c}'}(W_b,\mathfrak{h})_0.$$

Lemma 5.17. *The centre $Z_{\mathbf{c}}(W,\mathfrak{h})$ of $H_{\mathbf{c}}(\mathfrak{h},W)$ is a direct summand of $H_{\mathbf{c}}(\mathfrak{h},W)$ when considered as a $Z_{\mathbf{c}}(W,\mathfrak{h})$-module.*

Proof. First, let us show that $Z_{\mathbf{c}}(W,\mathfrak{h})$ is integrally closed. By Proposition 1.16 the skew group ring $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*] \rtimes W$ is a maximal order. The algebra $H_{\mathbf{c}}(\mathfrak{h},W)$ is $\mathbb{N}$-filtered and $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*] \rtimes W$ is its associated graded. Now [98, Theorem 5] shows that the property of being a maximal order lifts to $H_{\mathbf{c}}(\mathfrak{h},W)$. The centre of a maximal order is integrally closed, see [74, Proposition 5.1.10]. The statement of the Lemma now follows from:

Claim Let A be a prime $\mathbb{C}$-algebra, finite over its centre Z which is integrally closed. Then Z is a direct summand of A as a Z-module.

Proof of claim: The centre Z is a domain. Let $Q(Z)$ be the field of fractions of Z and $D = A \otimes_Z Q(Z)$. By Posner's Theorem [74, Theorem 13.6.5], D is a central simple algebra. If $\overline{Q(Z)}$ is the algebraic closure of $Q(Z)$ then

$$A \otimes_Z \overline{Q(Z)} = D \otimes_{Q(Z)} \overline{Q(Z)} \simeq \mathrm{Mat}_n(\overline{Q(Z)}), \text{ for some } n. \tag{5.6}$$

Therefore we have a trace map $\mathsf{tr} : A \otimes_Z \overline{Q(Z)} \to \overline{Q(Z)}$. It is shown in [42, page 38] that one can choose the isomorphism (5.6) so that $\mathsf{tr}_| : D \to Q(Z)$. Now choose $a \in A$. Since A is a finite module over Z there exists a monic polynomial $f \in Z[x]$ such that $f(a) = 0$. Let $g \in \overline{Q(Z)}[x]$ be the minimal polynomial of a, considered as an element of $\mathrm{Mat}_n(\overline{Q(Z)})$ and let the roots of g be $\alpha_1, \ldots, \alpha_k$. Since $g \,|\, f$ in $\overline{Q(Z)}[x]$, $f(\alpha_i) = 0$ for all roots α_i of g. Therefore the algebra $B := Z[\alpha_1, \ldots, \alpha_k]$ is a finite Z-module. The coefficients of g belong to B. In particular, $\mathsf{tr}(a) \in Q(Z) \cap B = Z$ since Z is assumed to be integrally closed. Therefore $\mathsf{tr}(A) = Z$. The map $\frac{1}{n}\mathsf{tr}$ is a Z-module morphism and its kernel is the required complement to Z in A. $\square$

Theorem 5.18. *Fix b an element of $\mathfrak{h}$ and let $\mathbf{c}'$ be the restriction of $\mathbf{c}$ to the subgroup W_b of W. There is a Poisson isomorphism*

$$\theta : \widehat{Z}_{\mathbf{c}}(W, \mathfrak{h})_b \xrightarrow{\sim} \widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{h})_0.$$

Proof. Lemma 5.17 and Proposition 5.16 show that the assumptions of (5.12) hold. Therefore the theorem follows from Proposition 5.12. $\square$

Remark 5.19. In Theorem 5.18 it is possible to choose a point $\lambda \in \mathfrak{h}^*/W$ instead of $b \in \mathfrak{h}/W$, the analogous statement holds.

Let us fix $\mathfrak{t} := (\mathfrak{h}^{*\,W_b})^{\perp} \subset \mathfrak{h}$ and $\mathfrak{s} := \mathfrak{h}^{W_b}$ so that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{s}$. The defining relations of $H_{t,\mathbf{c}}$ show that

$$H_{t,\mathbf{c}}(W_b, \mathfrak{h}) \simeq H_{t,\mathbf{c}}(W_b, \mathfrak{t}) \otimes \mathcal{D}_t(\mathfrak{s}). \tag{5.7}$$

Here, for a given vector space V, $\mathcal{D}_t(V)$ is the $\mathbb{C}$-algebra generated by V and V^*, the elements of V commuting amongst themselves and similarly for the elements of V^*, whilst $[x, y] = t \cdot x(y)$ for $y \in V$ and $x \in V^*$. Thus, when $t \neq 0$, $\mathcal{D}_t(V)$ is isomorphic to the Weyl algebra over V and when $t = 0$, $\mathcal{D}_t(V) = \mathbb{C}[V \times V^*]$. However $\mathbb{C}[V \times V^*]$ inherits a non-degenerate Poisson structure from $\mathcal{D}_t(V)$ given by $\{x, x'\} = \{y, y'\} = 0$ and $\{x, y\} = x(y)$, for $x, x' \in V^*$ and $y, y' \in V$, which is a particular case of the construction given in (1.3). Equivalently $V \times V^*$ is a symplectic manifold with the canonical symplectic structure. The isomorphism (5.7) restricts to an isomorphism of the centres. Moreover, since (5.7) is valid for all t, the isomorphism of centres is a Poisson isomorphism when $t = 0$. If $\widehat{\mathbb{C}}[\mathfrak{s} \times \mathfrak{s}^*]_0$ is the completion of the polynomial ring $\mathbb{C}[\mathfrak{s} \times \mathfrak{s}^*]$ with respect to the ideal generated by $\mathbb{C}[\mathfrak{s}]_+$ then there is an isomorphism of Poisson algebras

$$Z_{\mathbf{c}'}(W_b, \mathfrak{h}) \simeq Z_{\mathbf{c}'}(W_b, \mathfrak{t}) \otimes \mathbb{C}[\mathfrak{s} \times \mathfrak{s}^*], \tag{5.8}$$

which extends to an isomorphism of complete Poisson algebras

$$\widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{h})_0 \simeq \widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{t})_0 \,\widehat{\otimes}\, \widehat{\mathbb{C}}[\mathfrak{s} \times \mathfrak{s}^*]_0.$$

5.6 The Etingof-Ginzburg Sheaf

The goal of this section is to prove the factorization result (5.1) and Theorem 5.3. As a consequence of Theorem 5.14, we have an isomorphism of quotient algebras.

Proposition 5.20. *Let $\theta : \widehat{H}_{0,\mathbf{c}}(W,\mathfrak{h})_b \to C(W, W_b, \widehat{H}_{0,\mathbf{c}'}(W_b,\mathfrak{h})_0)$ be the isomorphism (5.3). Then θ descends to an isomorphism*

$$\theta : \frac{H_{0,\mathbf{c}}(W,\mathfrak{h})}{\langle \mathfrak{m}(b)\rangle} \xrightarrow{\sim} C\left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b,\mathfrak{h})}{\langle \mathfrak{n}(0)\rangle}\right). \tag{5.9}$$

Proof. Proposition 5.16 implies that

$$\theta(\mathfrak{m}(b)\widehat{H}_{t,\mathbf{c}}(W,\mathfrak{h})_b) = C(W, W_b, \mathfrak{n}(0)\widehat{H}_{t,\mathbf{c}'}(W_b,\mathfrak{h})_0),$$

and the isomorphism follows. □

Recall from (2.4) that the inclusion $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow Z_{\mathbf{c}}(W,\mathfrak{h})$ defines a surjective morphism $\pi_W := \pi_1 : X_{\mathbf{c}}(W) \twoheadrightarrow \mathfrak{h}/W$. The algebra $Z_{\mathbf{c}}(W,\mathfrak{h})/\langle\mathfrak{m}(b)\rangle$ is the coordinate ring of the scheme-theoretic pull-back $\pi_W^{-1}(b)$. Comparing the centres of the algebras in Proposition 5.20 gives an isomorphism of (non-reduced) schemes:

Corollary 5.21. *For $b \in \mathfrak{h}$, there is a scheme-theoretic isomorphism*

$$\Phi : \pi_W^{-1}(b) \xrightarrow{\sim} \pi_{W_b}^{-1}(0) \tag{5.10}$$

Proof. It is not clear that the centre of $H_{0,\mathbf{c}}(W,\mathfrak{h})/\langle\mathfrak{m}(b)\rangle$ equals $Z_{\mathbf{c}}(W,\mathfrak{h})/\langle\mathfrak{m}(b)\rangle$ (there is an example [18, Example 3.19] of an analogous situation of the enveloping algebra of a Lie algebra in positive characteristic where the centre of a quotient is greater than the corresponding quotient of the centre). To overcome this we use the Satake isomorphism, Theorem 2.5. Since $\mathfrak{m}(b)\, H_{0,\mathbf{c}}(W,\mathfrak{h})$ is a centrally generated ideal in $H_{0,\mathbf{c}}(W,\mathfrak{h})$,

$$\mathfrak{m}(b)\, H_{0,\mathbf{c}}(W,\mathfrak{h}) \cap \mathbf{e}_W\, H_{0,\mathbf{c}}(W,\mathfrak{h})\, \mathbf{e}_W = \langle \mathbf{e}_W \cdot \mathfrak{m}(b)\rangle,$$

where the right-hand side is considered as an ideal in $\mathbf{e}_W\, H_{0,\mathbf{c}}(W,\mathfrak{h})\, \mathbf{e}_W$. Therefore the Satake isomorphism descends to an isomorphism

$$\mathsf{S}_{W,b} \,:\, \frac{Z_{0,\mathbf{c}}(W)}{\mathfrak{m}(b) Z_{0,\mathbf{c}}(W)} \xrightarrow{\sim} \mathbf{e}_W \left(\frac{H_{0,\mathbf{c}}(W,\mathfrak{h})}{\langle\mathfrak{m}(b)\rangle}\right) \mathbf{e}_W. \tag{5.11}$$

As noted in [10, Lemma 3.1 (ii)], the isomorphism (5.9) restricts to an isomorphism of subalgebras

$$\theta \,:\, \mathbf{e}_W \left(\frac{H_{0,\mathbf{c}}(W,\mathfrak{h})}{\langle\mathfrak{m}(b)\rangle}\right) \mathbf{e}_W \xrightarrow{\sim} \mathbf{e}_{W_b} \left(\frac{H_{0,\mathbf{c}'}(W_b,\mathfrak{h})}{\langle\mathfrak{n}(0)\rangle}\right) \mathbf{e}_{W_b}, \tag{5.12}$$

where $\theta(\mathbf{e}_W) = \mathbf{e}_{W_b}$. Here we have identified the spherical subalgebra on the right-hand side with a subalgebra of $C\left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b,\mathfrak{h})}{\langle\mathfrak{n}(0)\rangle}\right)$. This is not the "standard" embedding. The precise description of this embedding has been moved to section A.2 of the appendix since the details are not important for us here. Combining the isomorphisms of (5.11) and (5.12) produces the comorphism

$$(\Phi^*)^{-1} = \mathsf{S}_{W_b,0}^{-1} \circ \theta \circ S_{W,b} \, : \, \frac{Z_{0,\mathbf{c}}(W)}{\mathfrak{m}(b) Z_{0,\mathbf{c}}(W)} \xrightarrow{\sim} \frac{Z_{0,\mathbf{c}'}(W_b)}{\mathfrak{n}(0) Z_{0,\mathbf{c}'}(W_b)} \tag{5.13}$$

corresponding to Φ. □

We now conclude:

Theorem 5.22. *Let $\mathcal{R}[W]$ be the Etingof-Ginzburg sheaf on $X_{\mathbf{c}}(W)$ and $\mathcal{R}[W_b]$ the Etingof-Ginzburg sheaf on $X_{\mathbf{c}'}(W_b)$. For $b \in \mathfrak{h}/W$ we have an isomorphism of W-equivariant sheaves on $\pi_{W_b}^{-1}(0)$,*

$$\Phi_* \left(\mathcal{R}[W]|_{\pi_W^{-1}(b)} \right) \simeq \mathit{Ind}_{W_b}^{W} \mathcal{R}[W_b]|_{\pi_{W_b}^{-1}(0)}. \tag{5.14}$$

Proof. Since $\pi_W^{-1}(b)$ is an affine scheme, to show that we have an isomorphism of W-equivariant sheaves as stated in (5.14) it suffices to show that the global sections are isomorphic as $(W, Z_{0,\mathbf{c}'}(W_b) \,/\, \langle \mathfrak{n}(0) \rangle =: \mathsf{Z})$-bimodules. Taking global sections gives

$$\Phi_* \left(\mathcal{R}[W]|_{\pi_W^{-1}(b)} \right) (\pi_{W_b}^{-1}(0)) \simeq \left(\frac{H_{0,\mathbf{c}}(W)}{\langle \mathfrak{m}(b) \rangle} \right) \mathbf{e}_W,$$

where the space on the right hand side becomes a Z-module via Φ^*, and

$$\mathtt{Ind}_{W_b}^{W} \mathcal{R}[W_b]|_{\pi_{W_b}^{-1}(0)} (\pi_{W_b}^{-1}(0)) = \mathtt{Ind}_{W_b}^{W} \left(\frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \mathbf{e}_{W_b}.$$

Thus we must show that

$$\mathsf{He} := \left(\frac{H_{0,\mathbf{c}}(W)}{\langle \mathfrak{m}(b) \rangle} \right) \mathbf{e}_W \simeq \mathtt{Ind}_{W_b}^{W} \left(\frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \mathbf{e}_{W_b}$$

as (W, Z)-bimodules. Applying the isomorphism θ (of (5.9)) to He, and noting that the restriction of θ to $\mathbb{C}W$ is the map ι, gives

$$\theta \,:\, \mathsf{He} \simeq C \left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \iota(\mathbf{e}_W).$$

However, we now have two different actions of Z on He. It acts on He, viewed as global sections, via the map Φ^*, but acts on the right of $C \left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \iota(\mathbf{e}_W)$ via θ^{-1}. These two actions are the same: as stated in (5.13),

$$\Phi^* = \mathsf{S}_{W,b}^{-1} \circ \theta^{-1} \circ S_{W_b,0},$$

therefore

$$h \, \mathbf{e}_W \cdot \Phi^*(z) = h \, \mathbf{e}_W \cdot \mathsf{S}_{W,b}^{-1} \circ \theta \circ S_{W_b,0}(z) = h \, \mathbf{e}_W \cdot \mathbf{e}_W \, \theta^{-1}(\mathbf{e}_{W_b} \cdot z) = h \, \mathbf{e}_W \cdot \theta^{-1}(z),$$

where $z \in \mathsf{Z}$ and $h \, \mathbf{e}_W \in \mathsf{He}$ (recall that $\theta(\mathbf{e}_W) = \mathbf{e}_{W_b}$ as in (5.13)). Noting that Z is a subalgebra of the centre of $H_{0,\mathbf{c}'}(W_b, \mathfrak{h}) \,/\, \langle \mathfrak{n}(0) \rangle$, the required bimodule isomorphism is given by Lemma 5.13 where $G = W$, $H = W_b$ and $A = H_{0,\mathbf{c}'}(W_b, \mathfrak{h}) \,/\, \langle \mathfrak{n}(0) \rangle$. □

Example 5.23. Recall that in the case $W = S_n, \mathfrak{h} = \mathbb{C}^n$ and $\mathbf{c} \neq 0$ the Calogero-Moser space $X_{\mathbf{c}}(S_n)$ has been shown to be smooth by Etingof and Ginzburg, [36, Corollary 16.2] or [102, Proposition 1.7]. Therefore [36, Theorem 1.7 (i)] implies that $\mathcal{R}[S_n]$ is a vector bundle of rank $n!$ on $X_{\mathbf{c}}(S_n)$. Identifying $\mathbb{C}^n/S_n$ with $S^n(\mathbb{C})$, a point of $\mathbb{C}^n/S_n$ has the form $n_1x_1 + \cdots + n_kx_k$, where $n_1 + \cdots + n_k = n$ and $x_1, \ldots, x_k \in \mathbb{C}$ are pairwise distinct. Given $b \in \mathbb{C}^n$ such that $S_n \cdot b = n_1x_1 + \cdots + n_kx_k$, the stabilizer $(S_n)_b$ is conjugate to $S_{n_1} \times \cdots \times S_{n_k}$. For $W = S_n$, the isomorphism of Corollary 5.21 induces, after factoring out nilpotent elements, an isomorphism of varieties

$$\pi_{S_n}^{-1}(b) \simeq \pi_{S_{n_1}}^{-1}(0) \times \cdots \times \pi_{S_{n_k}}^{-1}(0). \tag{5.15}$$

It is not clear that this isomorphism equals the one constructed by Wilson, Lemma 5.2. Let $\boxtimes$ denote the external tensor product of vector bundles, then Theorem 5.22 implies that

$$\Phi_* \left(\mathcal{R}[S_n]|_{\pi_{S_n}^{-1}(b)} \right) \simeq \mathtt{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_n} \left(\mathcal{R}[S_{n_1}]|_{\pi_{S_{n_1}}^{-1}(0)} \boxtimes \cdots \boxtimes \mathcal{R}[S_{n_k}]|_{\pi_{S_{n_k}}^{-1}(0)} \right)$$

as S_n-equivariant vector bundles. This confirms the conjectured factorization given in [36, 11.27].

5.7 Labeling symplectic leaves

In this section we will see that one can use the isomorphism of Theorem 5.18 to label every symplectic leaf in $X_{\mathbf{c}}(W)$ by a conjugacy class of parabolic subgroups of W. However there will, in general, be distinct leaves labeled by the same conjugacy class. Using this labeling we show that each leaf can be "induced" from a zero-dimensional leaf in the generalized Calogero-Moser space of a representative of the conjugacy classes that labels that leaf. This is analogous to the construction of Richardson orbits in Lie theory.

Fix a parabolic subgroup W_b of W and let $(\mathfrak{h}^{W_b})_{\text{reg}}$ be the set of points in $\mathfrak{h}$ whose stabilizer is W_b. The images of $\mathfrak{h}^{W_b}$ and $(\mathfrak{h}^{W_b})_{\text{reg}}$ in $\mathfrak{h}/W$ will be written $\mathfrak{h}^{(W_b)}/W$ and $\mathfrak{h}_{\text{reg}}^{(W_b)}/W$ respectively. They only depend on the conjugacy class of W_b. The sets $\mathfrak{h}_{\text{reg}}^{(W_b)}/W$ define a finite stratification of $\mathfrak{h}/W$ by locally closed subsets. Moreover, the closure ordering that this stratification defines agrees with the partial ordering on conjugacy classes of parabolic subgroups defined in (2.8):

$$(W_1) \geq (W_2) \quad \Longleftrightarrow \quad \mathfrak{h}_{\text{reg}}^{(W_2)} \, / \, W \subseteq \overline{\mathfrak{h}_{\text{reg}}^{(W_1)} \, / \, W}.$$

Lemma 5.24. *Let (W_b) be a conjugacy class of parabolic subgroups of W of rank r, then*

$$\dim \mathfrak{h}_{reg}^{(W_b)} \, / \, W = n - r.$$

Proof. Since $\mathfrak{h}_{\text{reg}}^{(W_b)}/W$ is an open subset of the irreducible variety $\mathfrak{h}^{(W_b)}/W$, $\dim \mathfrak{h}_{\text{reg}}^{(W)}/W = \dim \mathfrak{h}^{(W_b)}/W$. As explained in section (2.8), there is a W'-equivariant decomposition $\mathfrak{h} = \mathfrak{h}^{W_b} \oplus (\mathfrak{h}^{*W_b})^{\perp}$ with $\dim (\mathfrak{h}^{*W_b})^{\perp} = r$. Hence $\dim \mathfrak{h}^{(W_b)} = n - r$. Since the quotient map $\mathfrak{h} \twoheadrightarrow \mathfrak{h}/W$ is a finite surjective morphism, $\dim \mathfrak{h}^{(W_b)}/W = \dim \mathfrak{h}^{W_b} = n - r$. $\square$

Recall from (2.4) that we have surjective morphisms $\pi_1 : X_{\mathbf{c}}(W,\mathfrak{h}) \twoheadrightarrow \mathfrak{h}^*/W$ and $\pi_2 : X_{\mathbf{c}}(W,\mathfrak{h}) \twoheadrightarrow \mathfrak{h}/W$ defined by the inclusions $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow Z_{\mathbf{c}}(W,\mathfrak{h})$ and $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow Z_{\mathbf{c}}(W,\mathfrak{h})$ respectively. The map Υ was defined to be $\pi_1 \times \pi_2 : X_{\mathbf{c}}(W,\mathfrak{h}) \twoheadrightarrow \mathfrak{h}^*/W \times \mathfrak{h}/W$.

Proposition 5.25. *Let $\mathcal{L}$ be a symplectic leaf in $X_{\mathbf{c}}(W,\mathfrak{h})$ of dimension $2l$.*

1. *There exists a unique conjugacy class (W_p) of parabolic subgroups of W with rank $(W_p) = n - l$ such that*
$$\mathcal{L} \cap \pi_1^{-1}(\mathfrak{h}_{reg}^{(W_p)}/W) \neq \emptyset.$$
2. *There exists a unique conjugacy class (W_q) of parabolic subgroups of W with rank $(W_q) = n - l$ such that*
$$\mathcal{L} \cap \pi_2^{-1}(\mathfrak{h}_{reg}^{*(W_q)}/W) \neq \emptyset.$$

In general $(W_p) \neq (W_q)$.

Proof. Let P be the Poisson primitive ideal of $Z_{\mathbf{c}}(W,\mathfrak{h})$ defining the closure of $\mathcal{L}$ in $X_{\mathbf{c}}$. The map Υ is a closed, finite, surjective morphism, therefore $\Upsilon(\mathcal{L})$ is a locally closed set of dimension $2l$. It is contained in the locally closed set $\pi_1(\mathcal{L}) \times \pi_2(\mathcal{L}) \subseteq \mathfrak{h}/W \times \mathfrak{h}^*/W$. Therefore

$$\dim \pi_1(\mathcal{L}) + \dim \pi_2(\mathcal{L}) = \dim (\pi_1(\mathcal{L}) \times \pi_2(\mathcal{L})) \geq 2l.$$

This means that either $\dim \pi_1(\mathcal{L}) \geq l$ or $\dim \pi_2(\mathcal{L}) \geq l$. For now let us assume that $\dim \pi_1(\mathcal{L}) \geq l$. Choose a conjugacy class (W_b) of parabolic subgroups of minimal rank such that $\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W \cap \pi_1(\mathcal{L}) \neq \emptyset$. Minimality of the rank of (W_b) is equivalent to asking that the dimension of $\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W$ in $\mathfrak{h}/W$ is maximal with respect to the property $\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W \cap \pi_1(\mathcal{L}) \neq \emptyset$. Since the stratification of $\mathfrak{h}/W$ by the locally closed subsets $\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W$ is finite, the set $\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W \cap \pi_1(\mathcal{L})$ is open in $\pi_1(\mathcal{L})$. Denote by P' a prime ideal of $\widehat{Z}_{\mathbf{c}}(W,\mathfrak{h})_b$ that is minimal over the ideal $P \otimes_{Z_{\mathbf{c}}(W,\mathfrak{h})} \widehat{Z}_{\mathbf{c}}(W,\mathfrak{h})_b$. By Lemma 5.10, it is a Poisson primitive ideal. Let $\theta : \widehat{Z}_{\mathbf{c}}(W,\mathfrak{h})_b \xrightarrow{\sim} \widehat{Z}_{\mathbf{c}'}(W_b,\mathfrak{h})_0$ be the isomorphism of Theorem 5.18. Lemma 5.9 says that the ideal $Q' := \theta(P') \cap Z_{\mathbf{c}'}(W_b,\mathfrak{h})$ is a Poisson primitive ideal. The isomorphism (5.8) implies that

$$\mathrm{V}\,(Q') \simeq \overline{\mathcal{M}} \times \mathfrak{s} \times \mathfrak{s}^*, \tag{5.16}$$

where

$$\overline{\mathcal{M}} = \mathrm{V}\,(Q' \cap Z_{\mathbf{c}'}(W_b,\mathfrak{t})) \subset X_{\mathbf{c}'}(W_b,\mathfrak{t}),$$

is the closure of some symplectic leaf $\mathcal{M}$. Fix rank $(W_b) = r$. Let us try to calculate the dimension of $\mathcal{M}$. Lemma 5.24 says that $\dim \pi_1(\mathcal{L}) \leq n - r$. Lemmata 5.7 and 5.9 show that $\mathrm{ht}\,(Q') = \mathrm{ht}\,(P)$. Therefore

$$2l = \dim \mathcal{L} = 2n - \mathrm{ht}(P) = 2n - \mathrm{ht}(Q').$$

Since $\dim\, \mathfrak{s} \times \mathfrak{s}^* = 2(n-r)$, equation (5.16) shows that

$$\dim\, \mathcal{M} + 2(n-r) = 2n - \mathrm{ht}\,(Q') = 2l.$$

However $l \leq \dim \pi_1(\mathcal{L}) \leq n-r$ implies that $\dim \pi_1(\mathcal{L}) = l = n-r$ and $\dim \mathcal{M} = 0$. This also means that $\dim \pi_2(\mathcal{L}) = l$ and we could equally have chosen to work in $\mathfrak{h}^*/W$. Clearly

$$\pi_1(\mathcal{L}) \cap \mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W \neq \emptyset \iff \pi_1^{-1}(\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W) \cap \mathcal{L} \neq \emptyset.$$

The uniqueness statement of the proposition follows from the fact that $\overline{\pi_1(\mathcal{L})}$ is irreducible and that $\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W \cap \pi_1(\mathcal{L})$ is open and dense in $\pi_1(\mathcal{L})$. □

Let $W(\mathcal{L})$ denote the conjugacy class of parabolic subgroups of W associated to $\mathcal{L}$ by Proposition 5.25 (1). The partial ordering defined on the symplectic leaves of $X_{\mathbf{c}}$ by $\mathcal{L} \leq \mathcal{M}$ if and only if $W(\mathcal{L}) \leq W(\mathcal{M})$ in the ordering of (2.8) equals the partial ordering defined by the closure of the symplectic leaves.

Corollary 5.26. *Let $\mathcal{L}$ be a zero dimensional symplectic leaf in $X_{\mathbf{c}}(W, \mathfrak{h})$. Then $\mathcal{L} \subseteq \Upsilon^{-1}(0)$.*

Proof. Proposition 5.25 (1) implies that $\mathcal{L} \subset \pi_1^{-1}(0)$ and Proposition 5.25 (2) implies that $\mathcal{L} \subset \pi_2^{-1}(0)$, therefore $\mathcal{L} \subset \pi_1^{-1}(0) \cap \pi_2^{-1}(0) = \Upsilon^{-1}(0)$. □

Remark 5.27. It has been pointed out to the author by M. Martino that there is a direct proof of Corollary 5.26: The rational Cherednik algebra $H_{\mathbf{c}}(W, \mathfrak{h})$ is $\mathbb{Z}$-graded with $\deg x = 1$, $\deg y = -1$ and $\deg w = 0$ for $x \in \mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$, $y \in \mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$ and $w \in W$. The centre inherits a $\mathbb{Z}$-grading. Geometrically this says that there is an action of $\mathbb{C}^*$ on $X_{\mathbf{c}}(W, \mathfrak{h})$. The map Υ is $\mathbb{C}^*$-equivariant and it can be shown that 0 is the unique fixed point in $\mathfrak{h}/W \times \mathfrak{h}^*/W$. Since $\mathbb{C}^*$ is connected and the set $\Upsilon^{-1}(0)$ is finite, this is the set of $\mathbb{C}^*$-fixed points of $X_{\mathbf{c}}(W, \mathfrak{h})$. It is shown in [44, Remark 3.1] that there exists an element $\mathsf{eu} \in Z_{\mathbf{c}}(W, \mathfrak{h})$ (the "Euler operator"), such that $\{\mathsf{eu}, z\} = (\deg z) \cdot z$ for any homogeneous element $z \in Z_{\mathbf{c}}(W, \mathfrak{h})$. Therefore the infinitesimal action of $\mathbb{C}^*$ is given by the vector field $\{\mathsf{eu}, -\}$. Again using the fact that $\mathbb{C}^*$ is connected, we see that the fixed points of $X_{\mathbf{c}}(W, \mathfrak{h})$ correspond to those closed points whose maximal ideal is preserved by $\{\mathsf{eu}, -\}$. If $\mathcal{L}$ is zero-dimensional then the maximal ideal defining it is clearly preserved by $\{\mathsf{eu}, -\}$ and therefore $\mathcal{L} \subset \Upsilon^{-1}(0)$.

Lusztig and Spaltenstein [70] have shown that if $\mathfrak{g}$ is a semisimple Lie algebra (over $\mathbb{C}$), $\mathfrak{l}$ a Levi subalgebra of $\mathfrak{g}$ and $\mathcal{O}$ a nilpotent orbit (under the adjoint group L_{ad} of $\mathfrak{l}$) in $\mathfrak{l}^*$ then it is possible to "induce" $\mathcal{O}$ to a nilpotent orbit (under G_{ad}) in $\mathfrak{g}^*$. For an exposition of this construction see [28]. Motivated by this construction we try to do something similar, though to make the induction well-defined we are forced to consider the case of inducing from zero-dimensional leaves only. This is analogous to only considering Richardson orbits in the Lie theoretic picture.

Proposition 5.28. *Let (W_b), $b \in \mathfrak{h}$, be a conjugacy class of parabolic subgroups of W and choose a representative W_b of this class. Let $\mathcal{T}$ denote the set of all symplectic leaves $\mathcal{L}$ in $X_{\mathbf{c}}(W, \mathfrak{h})$ such that $W(\mathcal{L}) = (W_b)$. Then there exists a surjective map*

$$\mathtt{Ind}_{X_{\mathbf{c}'}(W_b)}^{X_{\mathbf{c}}(W)} : \{\textit{zero dimensional leaves of } X_{\mathbf{c}'}(W_b, \mathfrak{t})\} \twoheadrightarrow \mathcal{T},$$

though both sets may be empty (recall that $\mathfrak{t} = (\mathfrak{h}^{*W_b})^{\perp}$).

Proof. Symplectic leaves of $X_{\mathbf{c}}(W, \mathfrak{h})$ correspond to Poisson primitive ideals of $Z_{\mathbf{c}}(W, \mathfrak{h})$. Therefore we will define $\mathtt{Ind}$ in terms of Poisson primitive ideals. Since the closure $\mathfrak{h}^{(W_b)}/W$ of $\mathfrak{h}_{\text{reg}}^{(W_b)}/W$ in $\mathfrak{h}/W$ is irreducible, $\mathfrak{h}_{\text{reg}}^{(W_b)}/W$ is connected. Let $\mathcal{L} \in \mathcal{T}$. It was shown in the proof of Proposition 5.25 that

$$\dim \mathfrak{h}_{\text{reg}}^{(W_b)}/W \cap \pi_1(\mathcal{L}) = n - \operatorname{rank}(W_b) = \dim \mathfrak{h}_{\text{reg}}^{(W_b)}/W.$$

Therefore $\mathfrak{h}_{\text{reg}}^{(W_b)}/W \cap \pi_1(\mathcal{L})$ is open and dense in $\mathfrak{h}_{\text{reg}}^{(W_b)}/W$. Since the number of leaves in $\mathcal{T}$ is finite we can choose

$$b' \in \mathfrak{h}_{\text{reg}}^{(W_b)}/W \cap \bigcap_{\mathcal{L} \in \mathcal{T}} \pi_1(\mathcal{L}).$$

Without loss of generality we may assume $b' = b$. First we wish to show that there is a natural bijection between the set $\{\text{zero dimensional leaves of } X_{\mathbf{c}'}(W_b, \mathfrak{t})\} = \{\text{maximal and Poisson ideals of } Z_{\mathbf{c}'}(W_b, \mathfrak{t})\}$ and the set of Poisson primitive ideals of height $2 \operatorname{rank}(W_b)$ in $\widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{h})_0$. Let $\mathfrak{m}$ be a maximal and Poisson ideal of $Z_{\mathbf{c}'}(W_b, \mathfrak{t})$. The isomorphism (5.8) implies that the ideal generated by $\mathfrak{m}$ in $Z_{\mathbf{c}'}(W_b, \mathfrak{h})$ is a Poisson primitive ideal of height $2 \operatorname{rank}(W_b)$. Now set $Q = \mathfrak{m} \otimes_{Z_{\mathbf{c}}(W_b, \mathfrak{t})} \widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{h})_0$. It follows from Lemma 5.9 that Q is a Poisson ideal and every prime minimal over Q is Poisson primitive. Moreover, Lemma 5.7 (1) says that the height of each of these minimal primes is $2 \operatorname{rank}(W_b)$. Therefore it suffices to show that Q is itself prime. Noting that

$$Q = \left(\mathfrak{m} \otimes_{Z_{\mathbf{c}}(W_b, \mathfrak{t})} \widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{t})_0 \right) \otimes_{\widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{t})_0} \widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{h})_0,$$

repeated applications of Lemma 5.8 reduces the question to showing that $\mathfrak{m} \otimes_{Z_{\mathbf{c}}(W_b, \mathfrak{t})} \widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{t})_0$ is prime. But this follows from Lemma 5.7 (3), since Corollary 5.26 shows that the ideal generated in $Z_{\mathbf{c}}(W_b, \mathfrak{t})$ by the space $\mathfrak{n}(0)$ is contained in $\mathfrak{m}$. The definition of Ψ is now straight-forward: by Theorem 5.18 we may consider $Z_{\mathbf{c}}(W, \mathfrak{h})$ to be a subalgebra of $\widehat{Z}_{\mathbf{c}'}(W_b, \mathfrak{h})_0$ then

$$\mathtt{Ind}(\mathfrak{m}) := Z_{\mathbf{c}}(W, \mathfrak{h}) \cap Q.$$

Lemmata 5.7 and 5.9 show that $\mathtt{Ind}(\mathfrak{m})$ is a Poisson primitive ideal of height $2r$. The surjectivity of Ψ follows from the fact that each prime minimal over $P \otimes_{Z_{\mathbf{c}}(W, \mathfrak{h})} \widehat{Z}_{\mathbf{c}}(W, \mathfrak{h})_b$, $P \in \mathcal{T}$, corresponds to some zero dimensional leaf in $X_{\mathbf{c}'}(W_b, \mathfrak{t})$. $\square$

If $\mathbf{c} = 0$ then we recover a result by Brown and Gordon, [17, Proposition 7.7], removing the requirement that W be a Weyl group.

Corollary 5.29. *Let W be a complex reflection group, $\mathfrak{h}$ its reflection representation. Then the number of symplectic leaves of dimension $2l$ in $\mathfrak{h} \times \mathfrak{h}^* / W$ equals the number of conjugacy classes of parabolic subgroups of W of rank* $\dim \mathfrak{h} - l$.

Proof. Let W_b, $b \in \mathfrak{h}$ be a parabolic subgroup of W of rank r, $\mathfrak{t} \subset \mathfrak{h}$ its reflection representation. Then $\{0\}$ is the unique zero dimensional symplectic leaf in $\mathfrak{t} \times \mathfrak{t}^* / W_b$. Therefore Proposition 5.28 implies that there exists a unique symplectic leaf in $\mathfrak{h} \times \mathfrak{h}^* / W$ labeled by (W_b) and this leaf has dimension $2 \dim \mathfrak{h} - 2r$. $\square$

5.8 Cuspidal Representations

A closed point $\chi \in X_{\mathbf{c}}(W, \mathfrak{h})$ can be regarded as a non-zero algebra homomorphism $\chi : Z_{\mathbf{c}}(W, \mathfrak{h}) \to \mathbb{C}$. We define

$$H_{\mathbf{c},\chi} := \frac{H_{0,\mathbf{c}}(W, \mathfrak{h})}{\langle \operatorname{Ker} \chi \rangle},$$

a finite dimensional quotient of $H_{0,\mathbf{c}}(W, \mathfrak{h})$.

Definition 5.30. The algebra $H_{\mathbf{c},\chi}$ is said to be a *cuspidal algebra* if $\{\chi\}$ is a zero dimensional leaf of $X_{\mathbf{c}}$. A simple $H_{\mathbf{c}}(W, \mathfrak{h})$-module L is a *cuspidal representation* if L is a module for some cuspidal algebra $H_{\mathbf{c},\chi}$, or equivalently, Supp L is a zero dimensional symplectic leaf in $X_{\mathbf{c}}$.

Note that the space $X_{\mathbf{c}}(W, \mathfrak{h})$ may have no zero dimensional leaves. For instance, if $W = S_n, n > 1$ and $\mathbf{c} \neq 0$ then it is shown in [36, Corollary 1.14] that $X_{\mathbf{c}}$ is a symplectic manifold of dimension $2n$ and has no zero dimensional leaves.

5.9 Flows along symplectic leaves

The algebra $H_{\mathbf{c}}(\mathfrak{h}, W)$ can be considered as a sheaf of algebras on $X_{\mathbf{c}}(W, \mathfrak{h})$. The fiber of this sheaf at a point $\chi \in X_{\mathbf{c}}(W, \mathfrak{h})$ is $H_{\mathbf{c},\chi}$. Let $\mathcal{L}$ be a leaf in $X_{\mathbf{c}}$ and $\chi_1, \chi_2 \in \mathcal{L}$. Then we have the beautiful result [17, Theorem 4.2], based on [30, Corollary 9.2]:

$$\psi_{\chi_1,\chi_2} : H_{\mathbf{c},\chi_1} \xrightarrow{\sim} H_{\mathbf{c},\chi_2} \tag{5.17}$$

i.e. the representation theory of $H_{\mathbf{c}}(W, \mathfrak{h})$ is constant along the leaves of $X_{\mathbf{c}}(W, \mathfrak{h})$. We wish to show that this isomorphism is W-equivariant.

We recall here the construction of the isomorphism (5.17) as given in [17, Theorem 4.2]. Fix $H = H_{\mathbf{c}}(W, \mathfrak{h})$, $Z = Z_{\mathbf{c}}(W, \mathfrak{h})$ and let P be the Poisson prime defining the closure of $\mathcal{L}$. Then $H/P \cdot H$ is a Z/P-module and the algebras $H_{\mathbf{c},\chi_1}$ and $H_{\mathbf{c},\chi_2}$ are quotients of $H/P \cdot H$. The construction of (1.3) defines an action of $f \in Z$ on H as a derivation, $D_f(a) := \{f, a\}$ for $a \in H$. This makes H into a Poisson module for Z. By [17, Lemma 4.1], $H/P \cdot H$ is a Z/P-Poisson module with action induced from the derivations $D_f, f \in Z$. It is shown in the proof of [17, Theorem 4.2] that $H/P \cdot H$ is a locally free sheaf when restricted to $\mathcal{L}$. The space $\mathcal{L}$ is a smooth quasi-projective variety and we will now consider it as a complex analytic variety. Let $\hat{Z}$ be the algebra of holomorphic functions on $\mathcal{L}$ and define $\hat{H} = H \otimes_{(Z/P)} \hat{Z}$. The derivations D_f extend to derivations on $\hat{H}$ because the Poisson structure extends uniquely to $\hat{Z}$. For each point $\chi \in \mathcal{L}$, the natural map $H_{\mathbf{c},\chi} \to \hat{H}_\chi$ is an algebra isomorphism. Any two points χ_1 and χ_2 on $\mathcal{L}$ can be connected by a finite number of Hamiltonian flows: it is these flows that induce the isomorphism (5.17).

Therefore we may assume that there exists $f \in \hat{Z}$ and a Hamiltonian flow $\rho : B \to \mathcal{L}$ for f (where $B \subset \mathbb{C}$ is a small disk around zero) such that $\rho(0) = \chi_1$ and $\rho(t) = \chi_2$. Shrinking B if necessary and choosing an open neighbourhood U of $\rho(B)$ in $\mathcal{L}$, we may assume by Darboux's Theorem that we are in the following explicit situation: $U \subset \mathbb{C}^{2m}$ is an open, simply connected set containing χ_1, χ_2; $\mathcal{O}_U$ is the sheaf of holomorphic functions on U and $x_1, \dots, x_m, y_1, \dots, y_m$ are symplectic coordinates on U. That is, there is a non-degenerate Poisson bracket on $\mathcal{O}_U$ defined by $\{x_i, y_j\} = \delta_{ij}$ and $\{x_i, x_j\} = \{y_i, y_j\} = 0$ for all $1 \le i, j \le m$. Then $H' := \hat{H} \otimes_{\hat{Z}} Z'$ is an algebra containing $Z' = \mathcal{O}_U(U)$ such that $H' = \bigoplus_{i=1}^n Z' \cdot a_i$ is free as a Z'-module. The action of D_f on H' is defined by

$$D_f(x_i) = c_i(x, y), \quad D_f(y_j) = d_j(x, y) \quad \text{and} \quad D_f(a_i) = \sum_{j=1}^n e_{ij}(x, y) a_j,$$

for some functions $c_i, d_i, e_{ij} \in \mathcal{O}_U$. The algebra H' is the space of global sections of the trivial vector bundle $U \times \mathbb{C}^n$ over U. We fix coordinates $z_1, \dots, z_n$ on $\mathbb{C}^n$ such that $z_i(a_j) = \delta_{ij}$. Then the derivative D_f can be expressed explicitly as

$$D_f = \sum_{i=1}^m \left(c_i(x, y) \frac{\partial}{\partial x_i} + d_i(x, y) \frac{\partial}{\partial y_i} \right) - \sum_{i,j=1}^n e_{ji}(x, y) z_j \frac{\partial}{\partial z_i},$$

the minus sign appears because the z_i are dual to the a_i. The flow $\rho = (\rho_1, \dots, \rho_m, \rho'_1, \dots, \rho'_m)$ on U with respect to D_f satisfies $D_f(h)(\rho(t)) = \frac{d\rho}{dt}(h)(t)$ for all $h \in \mathcal{O}_U$ and is given explicitly as the solution to the system of equations

$$\frac{d\rho_i}{dt} = c_i(\rho(t)), \quad \frac{d\rho'_i}{dt} = d_i(\rho(t)), \quad 1 \le i \le m. \tag{5.18}$$

It is clear from the presentation that D_f actually defines a derivation of $\mathcal{O}_U[z_1, \dots, z_n]$. Every flow $\Psi : B \to U \times \mathbb{C}^n$ for D_f is a lift of a flow $\rho : B \to U$. This means that there exists some function $\psi : B \to \mathbb{C}^n$ such that $\Psi = (\rho, \psi)$. Explicitly, ψ satisfies the system of equations

$$\frac{d\psi_i}{dt} = - \sum_{j=1}^n e_{ji}(\rho(t)) \psi_j(t) \quad 1 \le i \le n. \tag{5.19}$$

Since this is a linear system of equations, the induced map on fibers $\psi_{\chi_1, \chi_2} : \hat{H}_{\chi_1} \to \hat{H}_{\chi_2}$ is linear. It is proved in [17, Theorem 4.2] that ψ_{χ_1, χ_2} is actually an algebra isomorphism.

Any section $w \in H'$ can be considered as a function $w \circ \rho : B \to U \times \mathbb{C}^n$ extending the flow ρ. Locally, there is a unique flow $\Psi : B \to U \times \mathbb{C}^n$ for D_f, lifting ρ and satisfying $\Psi(0) = w \circ \rho(0)$.

Lemma 5.31. *If $w \in H'$ such that $D_f(w) = 0$ then $\Psi = w \circ \rho$.*

Proof. By the uniqueness of flows it suffices to show that $w \circ \rho$ is a flow. Let us write $w = \sum_{i=1}^n g_i(x, y) a_i$. Then $D_f(w) = 0$ implies that

$$\sum_{i,j=1}^n \left(c_j(x, y) \frac{\partial g_i}{\partial x_j} + d_j(x, y) \frac{\partial g_i}{\partial y_j} \right) a_i + \sum_{i,j=1}^n g_i e_{ij}(x, y) a_j = 0,$$

hence

$$\sum_{j=1}^{n}\left(c_j(x,y)\frac{\partial g_i}{\partial x_j}+d_j(x,y)\frac{\partial g_i}{\partial y_j}\right)+\sum_{j=1}^{n} g_j e_{ji}(x,y)=0, \quad \forall\, 1\le i\le n. \tag{5.20}$$

Equation (5.19) shows that it suffices to prove that $\frac{d(g_i\circ\rho)}{dt}=-\sum_{j=1}^{n} e_{ji}(\rho(t))g_j\circ\rho(t)$. Using the chain rule, (5.18) and (5.20),

$$\frac{d(g_i\circ\rho)}{dt}=\sum_{j=1}^{n}\frac{\partial g_i}{\partial x_j}(\rho(t))\cdot\frac{d\rho_j}{dt}+\sum_{j=1}^{n}\frac{\partial g_i}{\partial y_j}(\rho(t))\cdot\frac{d\rho'_j}{dt}$$

$$=\sum_{j=1}^{n}\left(\frac{\partial g_i}{\partial x_j}(\rho(t))\cdot c_j(\rho(t))+\frac{\partial g_i}{\partial y_j}(\rho(t))\cdot d_j(\rho(t))\right)=-\sum_{j=1}^{n} g_j(\rho(t))e_{ji}(\rho(t)).$$

□

Corollary 5.32. *Let χ_1,χ_2 be points on the leaf $\mathcal{L}$. Then the algebra isomorphism $\psi_{\chi_1,\chi_2}: H_{\mathbf{c},\chi_1}\xrightarrow{\sim} H_{\mathbf{c},\chi_2}$ is W-equivariant.*

Proof. As explained above, the isomorphism ψ_{χ_1,χ_2} is the composition of finitely many isomorphisms induced from local Hamiltonian flows on $\mathcal{L}$. Therefore we may assume that we are in the explicit local situation described above. Let $w\in W$ and $a\in\hat{H}_{\chi_1}$. We wish to show that $\psi_{\chi_1,\chi_2}(w\cdot a)=w\cdot\psi_{\chi_1,\chi_2}(a)$. Since ψ_{χ_1,χ_2} is an algebra morphism this is equivalent to proving that $\psi_{\chi_1,\chi_2}(\bar{w})=\bar{w}$ where $\bar{w}$ is the image of w in $\hat{H}_{\chi_1}$ and $\hat{H}_{\chi_2}$ respectively. From the construction of the derivations D_f as given in (1.3) we see that $D_f(w)=0$ for all $f\in\hat{Z}$. In terms of the trivialization of $\hat{H}$ over U, $\bar{w}=w\circ\rho(0)\in\hat{H}_{\chi_1}$ and $\bar{w}=w\circ\rho(t)\in\hat{H}_{\chi_2}$ (where $t\in B$ such that $\rho(t)=\chi_2$). Thus the result is a consequence of Lemma 5.31. □

We can now state the main result of this section.

Theorem 5.33. *Let $\mathcal{L}$ be a leaf in $X_{\mathbf{c}}(W,\mathfrak{h})$ of dimension $2l$ and χ a point on $\mathcal{L}$. Then there exists a parabolic subgroup W_b, $b\in\mathfrak{h}$, of W of rank* $\dim\mathfrak{h}-l$ *and a cuspidal algebra $H_{\mathbf{c}',\psi}$ with $\psi\in X_{\mathbf{c}'}(W_b,\mathfrak{t})$ (recall that $\mathfrak{t}=(\mathfrak{h}^{W_b})^\perp$) such that*

$$H_{\mathbf{c},\chi}\simeq Mat_{|W/W_b|}(H_{\mathbf{c}',\psi}).$$

Proof. By Proposition 5.25 there exists a unique conjugacy class (W_b) of parabolic subgroups of W such that $\mathcal{L}\cap\pi_1^{-1}(\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W)\neq\emptyset$. Without loss of generality, $b\in\pi_1(\mathcal{L})\cap\mathfrak{h}_{\mathrm{reg}}^{(W_b)}/W$. Using the isomorphism (5.17) we may assume that $\chi\in\mathcal{L}\cap\pi_1^{-1}(b)$. Let $K=\mathrm{Ker}\,\chi$. Then $K\otimes_{Z_{\mathbf{c}}(W,\mathfrak{h})}\widehat{Z}_{\mathbf{c}}(W,\mathfrak{h})_b$ is a maximal ideal in $\widehat{Z}_{\mathbf{c}}(W,\mathfrak{h})_b\simeq\widehat{Z}_{\mathbf{c}'}(W_b,\mathfrak{h})_0$ and the arguments in the proof of Proposition 5.25 show that $M=Z_{\mathbf{c}'}(W_b,\mathfrak{t})\cap K$ is a maximal and Poisson ideal of $Z_{\mathbf{c}'}(W_b,\mathfrak{t})$. If $N=Z_{\mathbf{c}'}(W_b,\mathfrak{h})\cap K$ then the isomorphism (5.7) shows that

$$H_{0,\mathbf{c}'}(W_b,\mathfrak{h})/N\cdot H_{0,\mathbf{c}'}(W_b,\mathfrak{h})\simeq H_{0,\mathbf{c}'}(W_b,\mathfrak{t})/M\cdot H_{0,\mathbf{c}'}(W_b,\mathfrak{t})$$

is some cuspidal quotient $H_{\mathbf{c}',\psi}$ of $H_{0,\mathbf{c}'}(W_b,\mathfrak{t})$ (here Ker $\psi=M$). Now the isomorphism of Theorem 5.14 induces an isomorphism

$$\theta\,:\,H_{\mathbf{c},\chi}=\widehat{H}_{0,\mathbf{c}}(W,\mathfrak{h})_b/K\cdot\widehat{H}_{0,\mathbf{c}}(W,\mathfrak{h})_b\to C(W,W_b,\widehat{H}_{0,\mathbf{c}'}(W_b,\mathfrak{h})_0/N\cdot\widehat{H}_{0,\mathbf{c}'}(W_b,\mathfrak{h})_0)\simeq\mathrm{Mat}_{|W/W_b|}(H_{\mathbf{c}',\psi}).$$

$\square$

Remark 5.34. Let $H_{\mathbf{c},\chi}$ be a cuspidal algebra. Corollary 5.26 shows that there is a block B of the restricted rational Cherednik algebra $\bar{H}_{\mathbf{c}}(W)$ such that

$$H_{\mathbf{c},\chi} = \frac{B}{Z_{\mathbf{c}}(W) \cap B}.$$

In particular, every cuspidal module occurs as a simple module for the restricted rational Cherednik algebra.

Proposition 5.35. *Choose a point $\chi \in \mathcal{L}$ and let (W_b) be the conjugacy class of parabolic subgroups labeling $\mathcal{L}$ (as in Proposition 5.25 (1)). Then there exists a cuspidal algebra $H_{\mathbf{c}',\psi}$ for W_b and functor*

$$\Phi_{\psi,\chi} : H_{\mathbf{c}',\psi}\text{-mod} \xrightarrow{\sim} H_{\mathbf{c},\chi}\text{-mod}$$

defining an equivalence of categories such that

$$\Phi_{\psi,\chi}(M) \simeq \mathit{Ind}_{W_b}^{W} M \quad \forall\, M \in H_{\mathbf{c}',\psi}\text{-mod}$$

as W-modules.

Proof. If M is any $\widehat{H}_{0,\mathbf{c}'}(W_b, \mathfrak{t})_0$-module and θ the isomorphism of Theorem 5.14, then $\theta^*(M) = \mathrm{Fun}_{W_b}(W, M)$. As a W-module, $\mathrm{Fun}_{W_b}(W, M) \simeq \mathsf{Ind}_{W_b}^{W} M$. Taking $\chi' \in \pi^{-1}(b) \cap \mathcal{L}$ and fixing an isomorphism $\phi_{\chi',\chi} : H_{\mathbf{c},\chi'} \xrightarrow{\sim} H_{\mathbf{c},\chi}$ as in (5.17) defines an equivalence $(\phi_{\chi',\chi})_* : H_{\mathbf{c},\chi'}\text{-mod} \xrightarrow{\sim} H_{\mathbf{c},\chi}\text{-mod}$. Corollary 5.31 says that $\phi_{\chi',\chi}$ is W-equivariant therefore $\Phi_{\psi,\chi} = (\phi_{\chi',\chi})_* \circ \theta^*$ has the desired property. $\square$

Example 5.36. Let $I_2(m) = \langle a, b, \,|\, a^m = b^2 = 1, bab = a^{-1}\rangle$ be the dihedral group of order $2m$. When m is odd there is only one conjugacy class of reflections, $\{a^s b \,|\, 0 \le s \le m-1\}$, and when m is even there are two, $C_1 = \{a^s b \,|\, 0 \le s \le m-1, s \text{ even}\}$ and $C_2 = \{a^s b \,|\, 0 \le s \le m-1, s \text{ odd}\}$. The dihedral groups are rank two reflection groups therefore $\dim X_{\mathbf{c}}(I_2(m)) = 4$. For $m \ge 5$, it is always a singular variety as shown in [48, Proposition 7.3]. The conjugacy classes of parabolic subgroups in $I_2(m)$ are (1), $(\langle b\rangle)$ and $(I_2(m))$ when m is odd and (1), $(\langle b\rangle)$, $(\langle ab\rangle)$ and $(I_2(m))$ when m is even. By making use of Corollary 5.26 and knowing the blocks of the restricted rational Cherednik algebra, which are calculated in (6.10), one can show that the symplectic leaves for $X_{\mathbf{c}}(I_2(m))$ are described as in the tables (5.36).

In all cases, if χ is a point on a two dimensional leaf then $H_{\mathbf{c},\chi}$ is isomorphic to six by six matrices over the cuspidal algebra $H_{0,0}(S_2) = \mathbb{C}[x,y] \rtimes S_2/(x^2, xy, y^2)$. When $m = 6$, $I_2(6)$ is the Weyl group G_2. In this case, the cuspidal algebra supported on the zero dimensional leaf is a quotient of the algebra described in [36, Remark 16.5 (i)].

5.10 Remarks

1. The proof of the factorization of the generalized Calogero-Moser space and the corresponding factorization of the Etingof-Ginzburg sheaf has been published in the article [6]. The other results of the chapter have appeared in the preprint [5].

Table 5.1: Label, dimension and number of leaves for $I_2(m)$, m even

label	dim	# of leaves $\mathbf{c}=0$	$\mathbf{c}\in\{0\}\times\mathbb{C}^\times$	$\mathbf{c}\in\mathbb{C}^\times\times\{0\}$	$\mathbf{c}$ generic
(1)	4	1	1	1	1
$(\langle b\rangle)$	2	1	1	0	0
$(\langle ab\rangle)$	2	1	0	1	0
$(I_2(m))$	0	1	1	1	1

Table 5.2: Label, dimension and number of leaves for $I_2(m)$, m odd

label	dim	# of leaves $\mathbf{c}=0$	$\mathbf{c}\neq 0$
(1)	4	1	1
$(\langle b\rangle)$	2	1	0
$(I_2(m))$	0	1	1

Chapter 6

Calogero-Moser partitions

The goal of this chapter is to calculate the Calogero-Moser partition for the groups $G(m,d,n)$. The blocks of the restricted rational Cherednik algebra define a natural partitioning of the set of all simple $\bar{H}_{\mathbf{c}}(W)$-modules. Since the set of all simple modules can be identified with the set $\mathsf{Irr}(W)$ (2.5), the blocks of $\bar{H}_{\mathbf{c}}(W)$ define a partition of $\mathsf{Irr}(W)$, the Calogero-Moser partition. As explained in (2.6) the equivalence classes in the Calogero-Moser partition can be identified with the closed points in $\Upsilon^{-1}(0)$. These points are precisely the $\mathbb{C}^*$-fixed points on $X_{\mathbf{c}}(W)$, where the $\mathbb{C}^*$-action is the one described in remark 5.27.

When $W = G(m,1,n)(= C_m \wr S_n)$ and the parameter $\mathbf{c}$ is generic, Etingof and Ginzburg described the Calogero-Moser space as a certain affine quiver variety associated to the cyclic quiver with m vertices. It was shown by Martino [72] that this description is actually valid for all parameters. It is also possible to construct a symplectic resolution $Y_{\mathbf{c}}$ of $\mathbb{C}^{2n}/G(m,1,n)$ as a quiver variety associated to the same quiver (here $\mathbf{c}$ becomes a stability condition for the G.I.T. quotient). As noted by Gordon [49, §3], both these spaces have a hyperkähler structure and "rotating the hyperkähler structure" defines a map (of real spaces) $Y_{\mathbf{c}} \to X_{\mathbf{c}}$. This map is a diffeomorphism when the spaces are smooth and, by [49, Lemma 3.7], is $\mathbb{C}^*$-equivariant for some naturally defined action on $Y_{\mathbf{c}}$. Based on work by Gordon [49], he and Martino [51] used these facts to give a combinatorial description of the Calogero-Moser partition for $G(m,1,n)$. The aim of this chapter is to use Clifford theoretic arguments to extend this combinatorial description to all groups $G(m,d,n)$.

6.1 Blocks of normal subgroups

Throughout this section we fix an irreducible complex reflection group W with reflection representation $\mathfrak{h}$. Moreover we assume that there exists a normal subgroup $K \triangleleft W$ such that K also acts (via inclusion in W) on $\mathfrak{h}$ as a complex reflection group and $W/K \cong C_d$, the cyclic group of order d. Since K is normal in W, the group W acts on $\mathcal{S}(K)$ by conjugation. Let us fix a W-equivariant function $\mathbf{c} : \mathcal{S}(K) \to \mathbb{C}$. We extend this to a W-equivariant function $\mathbf{c} : \mathcal{S}(W) \to \mathbb{C}$ by setting $\mathbf{c}(s) = 0$ for $s \in \mathcal{S}(W) \backslash \mathcal{S}(K)$. Note that the partition of $\mathcal{S}(K)$ into K-orbits can be finer than the corresponding partition into W-

orbits. Thus a K-equivariant function on $\mathcal{S}(K)$ is not always W-equivariant. However, as will be shown below, this problem does not occur in the cases we consider. For our choice of parameter $\mathbf{c}$, the defining relations (2.2) show that the natural map $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes K \to H_{t,\mathbf{c}}(W)$ descends to an algebra morphism $H_{t,\mathbf{c}}(K) \to H_{t,\mathbf{c}}(W)$. The PBW property (2.1) shows that this map is injective.

Proposition 6.1. *For $\mathbf{c}$ as defined above, the algebra $H_{t,\mathbf{c}}(K)$ is a subalgebra of $H_{t,\mathbf{c}}(W)$.*

As explained above the goal of this chapter is to relate the Calogero-Moser partition of K to the Calogero-Moser partition of W. However the algebra $\bar{H}_{\mathbf{c}}(K)$ is not a subalgebra of $\bar{H}_{\mathbf{c}}(W)$. To overcome this we study an intermediate algebra, $\widetilde{H}_{\mathbf{c}}(K)$, which is defined to be the image of $H_{\mathbf{c}}(K)$ in $\bar{H}_{\mathbf{c}}(W)$. Thus we are in the following setup:

$$\begin{array}{ccc} H_{0,\mathbf{c}}(K) & \longrightarrow & H_{0,\mathbf{c}}(W) \\ \downarrow & & \downarrow \\ \widetilde{H}_{\mathbf{c}}(K) & \longrightarrow & \bar{H}_{\mathbf{c}}(W) \\ \downarrow & & \\ \bar{H}_{\mathbf{c}}(K) & & \end{array}$$

where the horizontal arrows are inclusions. To be precise, $\widetilde{H}_{\mathbf{c}}(K) := H_{0,\mathbf{c}}(K)/A_+ \cdot H_{0,\mathbf{c}}(K)$, where $A = \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ and A_+ the ideal of polynomials with constant term zero. The PBW theorem, Theorem 2.1, implies that $\widetilde{H}_{\mathbf{c}}(K) \cong \mathbb{C}[\mathfrak{h}]^{coW} \otimes \mathbb{C}K \otimes \mathbb{C}[\mathfrak{h}^*]^{coW}$ and hence has dimension $|K| \cdot |W|^2$. The idea is to relate the block partition of $\widetilde{H}_{\mathbf{c}}(K)$ to $\mathsf{CM}_{\mathbf{c}}(W)$ via the formalism of twisted symmetric algebras. The Proposition below shows that this allows us to deduce information about the partition $\mathsf{CM}_{\mathbf{c}}(K)$.

As noted in (2.5), the set $\{L(\lambda) \mid \lambda \in \mathsf{Irr}(K)\}$ is a complete set of non-isomorphic simple modules for $\bar{H}_{\mathbf{c}}(K)$. There is a natural surjective map $\widetilde{H}_{\mathbf{c}}(K) \twoheadrightarrow \bar{H}_{\mathbf{c}}(K)$ and the kernel of this map is generated by certain central nilpotent elements in $\widetilde{H}_{\mathbf{c}}(K)$. Therefore the kernel is contained in the radical of $\widetilde{H}_{\mathbf{c}}(K)$. This implies that $\{L(\lambda) \mid \lambda \in \mathsf{Irr}(K)\}$ is also a complete set of non-isomorphic simple modules for $\widetilde{H}_{\mathbf{c}}(K)$ and the block partition of $\widetilde{H}_{\mathbf{c}}(K)$ corresponds to a partition of the set $\mathsf{Irr}(K)$. In particular, the space $L(\lambda)$ is both a simple $\bar{H}_{\mathbf{c}}(K)$ and $\widetilde{H}_{\mathbf{c}}(K)$-module. However when we wish to consider $L(\lambda)$ as a $\widetilde{H}_{\mathbf{c}}(K)$-module we will denote it by $\tilde{L}(\lambda)$. For a given $\bar{H}_{\mathbf{c}}(K)$-module M we denote by $(M : L(\lambda))$ the multiplicity of $L(\lambda)$ in a composition series for M.

Proposition 6.2. *The Calogero-Moser partition $\mathsf{CM}_{\mathbf{c}}(K)$ of $\mathsf{Irr}(K)$ and the block partition of $\widetilde{H}_{\mathbf{c}}(K)$ on $\mathsf{Irr}(K)$ are equal because the blocks of $\widetilde{H}_{\mathbf{c}}(K)$ are the preimages of the blocks of $\bar{H}_{\mathbf{c}}(K)$ under the natural map $\widetilde{H}_{\mathbf{c}}(K) \twoheadrightarrow \bar{H}_{\mathbf{c}}(K)$.*

Proof. Let us again denote by A the algebra $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$ and define $B = \mathbb{C}[\mathfrak{h}]^K \otimes \mathbb{C}[\mathfrak{h}^*]^K$. Then we have inclusions $A \subset B \subset Z(H_{\mathbf{c}}(K)) \subset H_{\mathbf{c}}(K)$. The Proposition will follow from an application of a result of B. Müller; the version which we use here is stated in [18, Proposition 2.7]. Assume we are given an embedding of affine commutative $\mathbb{C}$-algebras $R \hookrightarrow Z$ such that Z is a finite R-module and there exists a prime $\mathbb{C}$-algebra T such that its centre is Z, over which it is a finite module. Then Müller's Theorem says that, for each maximal ideal $\mathfrak{m}$ of R, the primitive central idempotents of $T/\mathfrak{m}T$ are the images of the primitive idempotents of $Z/\mathfrak{m}Z$. Let us take $\mathfrak{m}_1 = A_+$, the maximal ideal of elements with constant term zero in A, $\mathfrak{m}_2 = B_+$, the maximal ideal of elements with constant term zero in B, $Z = Z(H_{\mathbf{c}}(K))$ and $T = H_{\mathbf{c}}(K)$. Then the primitive central idempotents of $T/\mathfrak{m}_1T$ are the images of the primitive idempotents of $Z/\mathfrak{m}_1$, and similarly for $T/\mathfrak{m}_2T$ and $Z/\mathfrak{m}_2$. However $\mathfrak{m}_1Z \subset \mathfrak{m}_2Z$ and $\mathfrak{m}_2Z/\mathfrak{m}_1Z$ is a nilpotent ideal in $Z/\mathfrak{m}_1Z$; therefore the primitive idempotents of $Z/\mathfrak{m}_2Z$ are the images of the primitive idempotents of $Z/\mathfrak{m}_1Z$. This implies that the primitive central idempotents of $T/\mathfrak{m}_2T$ are the images of the primitive central idempotents of $T/\mathfrak{m}_1T$. This is equivalent to the statement of the Proposition. $\square$

The following lemma will be required later.

Lemma 6.3. *We can choose a set $\{f_1, \ldots, f_n\}$ of homogeneous, algebraically independent generators of $\mathbb{C}[\mathfrak{h}]^K$ and positive integers $a_1, \ldots, a_n$ such that $\{f_1^{a_1}, \ldots, f_n^{a_n}\}$ is a set of homogeneous, algebraically independent generators of $\mathbb{C}[\mathfrak{h}]^W$ with $a_1 \cdots a_n = d$.*

Proof. The ring $\mathbb{C}[\mathfrak{h}]^K$ is $\mathbb{N}$-graded with $(\mathbb{C}[\mathfrak{h}]^K)_0 = \mathbb{C}$. Therefore $\mathfrak{m} := \mathbb{C}[\mathfrak{h}]^K_+$, the ideal of polynomials with zero constant terms, is the unique maximal, graded ideal of $\mathbb{C}[\mathfrak{h}]^K$. The group W acts on $\mathfrak{m}$ and hence also on $\mathfrak{m}^2$. Let U be a homogeneous, W-stable compliment to $\mathfrak{m}^2$ in $\mathfrak{m}$. By [9, Lemme 2.1], U generates $\mathbb{C}[\mathfrak{h}]^K$ and so $\mathbb{C}[\mathfrak{h}]^K = \mathbb{C}[U^*]$. The action of W on U^* factors through C_d. Since $\mathbb{C}[U^*]^{C_d} = \mathbb{C}[\mathfrak{h}]^W$ is a polynomial ring, the Chevalley-Shephard-Todd theorem, Theorem 1.12, says that C_d acts on U^* as a complex reflection group. Therefore we can decompose U into a direct sum of one-dimensional, homogeneous C_d-modules, $U = \oplus_{i=1}^n \mathbb{C} \cdot f_i$, and $C_d = C_{a_1} \times \cdots \times C_{a_n}$ such that the action of C_d on $\mathbb{C} \cdot f_i$ factors through C_{a_i} (and the action of C_{a_i} on $\mathbb{C} \cdot f_i$ is faithful). Then $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[f_1^{a_1}, \ldots, f_n^{a_n}]$ and the fact that $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial ring in n variables means that the polynomials $f_1^{a_1}, \ldots, f_n^{a_n}$ are algebraically independent. $\square$

Remark 6.4. For $W = G(m, 1, n)$ and $K = G(m, d, n)$ we can make an explicit choice of invariant polynomials as described in Lemma 6.3. Let $e_i(x_1, \ldots, x_n)$ denote the i^{th} elementary symmetric polynomial in $x_1, \ldots, x_n$. By [26, page 387], the following are a choice of algebraically independent homogeneous generators for $\mathbb{C}[\mathfrak{h}]^W$:

$$e_i(x_1^m, \ldots, x_n^m), \quad 1 \le i < n \qquad \text{and} \qquad (x_1 \ldots x_n)^{mn}.$$

In Lemma 6.3 we take f_n to be $(x_1 \ldots x_n)^{\frac{nm}{d}}$ and $f_i = e_i(x_1^m, \ldots, x_n^m)$ for $1 \le i < n$ so that $a_i = 1$ for $1 \le i < n$ and $a_n = d$.

6.2 Baby Verma modules

The results of this section are not required in the other parts of the thesis but they shed a little more light on the relationship between $\bar{H}_{\mathbf{c}}(K)$ and $\widetilde{H}_{\mathbf{c}}(K)$, as well as giving a representation theoretic proof of Proposition 6.2, so we have included them none the less. Just as we defined baby Verma modules and dual baby Verma modules for the restricted rational Cherednik algebra $\bar{H}_{\mathbf{c}}(W)$ we can do the same for the algebra $\widetilde{H}_{\mathbf{c}}(K)$.

Definition 6.5. Let $\lambda \in \mathsf{Irr}(K)$. The baby Verma module of $\widetilde{H}_{\mathbf{c}}(K)$ associated to λ is defined to be

$$\tilde{\Delta}(\lambda) := \widetilde{H}_{\mathbf{c}}(K) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes K} \lambda,$$

where $\mathbb{C}[\mathfrak{h}^*]^{coW}_+$ acts on λ as zero.

If N is a right $\widetilde{H}_{\mathbf{c}}(K)$ then N^* becomes a left $\widetilde{H}_{\mathbf{c}}(K)$- module, where the action of $\bar{H}_{\mathbf{c}}(K)$ on M^* is defined to be

$$(h \cdot f)(m) := f(m \cdot h) \quad \forall\, h \in \widetilde{H}_{\mathbf{c}}(K), m \in N, f \in N^*.$$

We use this fact to define dual baby Verma modules.

Definition 6.6. Let $\lambda \in \mathsf{Irr}(K)$. The *dual baby Verma module* of $\widetilde{H}_{\mathbf{c}}(K)$ associated to λ is

$$\tilde{\nabla}(\lambda) := \left(\lambda^* \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \rtimes K} \widetilde{H}_{\mathbf{c}}(K)\right)^*,$$

where $\mathbb{C}[\mathfrak{h}]^{coW}_+$ acts on λ^* as zero.

Recall (2.5) that we denote by $P(\lambda)$ the projective cover of $L(\lambda)$ in $\bar{H}_{\mathbf{c}}(K)$-mod. A $\bar{H}_{\mathbf{c}}(K)$-module M is said to have a Δ-filtration if there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of M by $\bar{H}_{\mathbf{c}}(K)$-submodules such that for each $i \geq 1$ there exists a $\lambda_i \in \mathsf{Irr}(K)$ with $M_i/M_{i-1} \cong \Delta(\lambda_i)$. For a given Δ-filtration of M we denote by $[M : \Delta(\lambda)]$ the number of $i \in \{1, \ldots, n\}$ such that $M_i/M_{i-1} \cong \Delta(\lambda)$. The result [56, Corollary 4.3] shows that this number is independent of the chosen filtration. Similarly the projective cover of $\tilde{L}(\lambda)$ will be denoted $\tilde{P}(\lambda)$ and we can talk of modules with $\tilde{\Delta}$-filtrations. For a $\widetilde{H}_{\mathbf{c}}(K)$-module N with a $\tilde{\Delta}$-filtration, the number $[M : \tilde{\Delta}(\lambda)]$ is independent of the chosen filtration. As an application of [56, Theorem 4.5] we have the following Brauer-type reciprocity result.

Theorem 6.7. *Any projective $\bar{H}_{\mathbf{c}}(K)$-module has a Δ-filtration. In particular, for $\lambda, \mu \in \mathsf{Irr}(K)$, the projective cover $P(\lambda)$ of $L(\lambda)$ has a Δ-filtration and*

$$[P(\lambda) : \Delta(\mu)] = (\nabla(\mu) : L(\lambda)).$$

Similarly, any projective $\widetilde{H}_{\mathbf{c}}(K)$-module has a $\tilde{\Delta}$-filtration and for $\lambda, \mu \in \mathsf{Irr}(K)$, the projective cover $\tilde{P}(\lambda)$ of $\tilde{L}(\lambda)$ has a $\tilde{\Delta}$-filtration with

$$[\tilde{P}(\lambda) : \tilde{\Delta}(\mu)] = (\tilde{\nabla}(\mu) : \tilde{L}(\lambda)).$$

Proposition 6.8. *Fix $\lambda \in \mathsf{Irr}(K)$. The $\widetilde{H}_{\mathbf{c}}(K)$-module $\tilde{\Delta}(\lambda)$ has a submodule filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_d = \tilde{\Delta}(\lambda)$ such that each $M_i/M_{i-1} \cong \Delta(\lambda)$ is actually a $\bar{H}_{\mathbf{c}}(K)$-module. Similarly, the $\widetilde{H}_{\mathbf{c}}(K)$-module $\tilde{\nabla}(\lambda)$ has a submodule filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_d = \tilde{\nabla}(\lambda)$ such that each $N_i/N_{i-1} \cong \nabla(\lambda)$ is actually a $\bar{H}_{\mathbf{c}}(K)$-module.*

Proof. Let $U \subset \mathbb{C}[\mathfrak{h}]^K$ be the subspace described in the proof of Lemma 6.3. It was shown that $\mathbb{C}[U^*]^{co\,C_d} = \mathbb{C}[\mathfrak{h}]^K / \langle \mathbb{C}[\mathfrak{h}]^W_+ \rangle$, where $\mathbb{C}[\mathfrak{h}]^W_+$ is the ideal of polynomials in $\mathbb{C}[\mathfrak{h}]^W$ with zero constant term. By [59, Proposition 3.6], $\mathbb{C}[\mathfrak{h}]$ is a free $\mathbb{C}[\mathfrak{h}]^K$-module of rank $|K|$. Therefore $\mathbb{C}[\mathfrak{h}]^{co\,W} = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^K}$

$\mathbb{C}[\mathfrak{h}]^K/\langle\mathbb{C}[\mathfrak{h}]^W_+\rangle$ is a free $\mathbb{C}[U^*]^{co\,C_d}$-module of rank $|K|$. The space $\mathbb{C}[U^*]^{coC_d}$ is $\mathbb{N}$-graded. Let V_i denote the component of degree i and $V_{>i}$ the sum $\bigoplus_{j>i} V_j$. Let l be the degree of the non-zero component of $\mathbb{C}[U^*]^{co\,C_d}$ of highest degree. We denote by $\mathbb{C}$ the trivial $\mathbb{C}[U^*]^{co\,C_d}$-module. Then, for each $1 \leq i \leq l-1$, we have a short exact sequence of $\mathbb{C}[U^*]^{co\,C_d}$-modules

$$0 \to V_{>i} \to V_{>i-1} \to \mathbb{C}^{\oplus \dim(V_i)} \to 0. \tag{6.1}$$

Note that $\mathbb{C}[\mathfrak{h}]^K_+ \cdot \mathbb{C}[\mathfrak{h}]^{co\,W} \otimes_{\mathbb{C}[U^*]^{co\,C_d}} \mathbb{C} = 0$ hence we get a surjective map

$$\mathbb{C}[\mathfrak{h}]^{co\,K} \twoheadrightarrow \mathbb{C}[\mathfrak{h}]^{co\,W} \otimes_{\mathbb{C}[U^*]^{co\,C_d}} \mathbb{C}.$$

The dimension of these spaces is equal therefore the map is actually an isomorphism. Tensoring the short exact sequence (6.1) by the free $\mathbb{C}[U^*]^{co\,C_d}$-module $\mathbb{C}[\mathfrak{h}]^{co\,W}$ produces the short exact sequence of $\mathbb{C}[\mathfrak{h}]^{co\,W} \rtimes K$-modules

$$0 \to V_{>i} \otimes_{\mathbb{C}[U^*]^{co\,C_d}} \mathbb{C}[\mathfrak{h}]^{co\,W} \to V_{>i-1} \otimes_{\mathbb{C}[U^*]^{co\,C_d}} \mathbb{C}[\mathfrak{h}]^{co\,W} \to (\mathbb{C}[\mathfrak{h}]^{co\,K})^{\oplus \dim(V_i)} \to 0. \tag{6.2}$$

We choose W-stable lifts V_i' of V_i in $\mathbb{C}[\mathfrak{h}]^K$ and write $V_{>i}' := \bigoplus_{j>i} V_j'$. The image of V_i' is central in $\widetilde{H}_{\mathbf{c}}(K)$. Therefore the short exact sequence (6.2) defines a short exact sequence of $\widetilde{H}_{\mathbf{c}}(K)$-modules

$$0 \to V_{>i}' \cdot \tilde{\Delta}(\lambda) \to V_{>i-1}' \cdot \tilde{\Delta}(\lambda) \to \Delta(\lambda)^{\oplus \dim(V_i)} \to 0,$$

for $1 \leq i \leq l-1$. One can now construct the filtration $0 = M_0 \subset M_1 \subset \dots$ by taking a suitable refinement of the above filtration of $\tilde{\Delta}(\lambda)$.
The corresponding result for $\tilde{\nabla}(\lambda)$ can be proved by repeating the above argument for the right $\widetilde{H}_{\mathbf{c}}(K)$-module $\lambda^* \otimes_{\mathbb{C}[\mathfrak{h}]^{coW} \rtimes K} \widetilde{H}_{\mathbf{c}}(K)$ and then dualizing. □

Proposition 6.9. *Let $\lambda, \mu \in \mathsf{Irr}(K)$, then*

$$\dim Hom_{\widetilde{H}_{\mathbf{c}}(K)}(\tilde{P}(\lambda), \tilde{P}(\mu)) = d^2 \dim Hom_{\bar{H}_{\mathbf{c}}(K)}(P(\lambda), P(\mu)).$$

Proof. Applying Theorem 6.7 and Proposition 6.8,

$$\begin{aligned}
\dim \operatorname{Hom}_{\widetilde{H}_{\mathbf{c}}(K)}(\tilde{P}(\lambda), \tilde{P}(\mu)) &= (\tilde{P}(\mu) : \tilde{L}(\lambda)) \\
&= \textstyle\sum_{\eta \in \mathsf{Irr}(K)} [\tilde{P}(\mu) : \tilde{\Delta}(\eta)](\tilde{\Delta}(\eta) : \tilde{L}(\lambda)) \\
&= \textstyle\sum_{\eta \in \mathsf{Irr}(K)} (\tilde{\nabla}(\eta) : \tilde{L}(\mu))(\tilde{\Delta}(\eta) : \tilde{L}(\lambda)) \\
&= d^2 \textstyle\sum_{\eta \in \mathsf{Irr}(K)} (\nabla(\eta) : L(\mu))(\Delta(\eta) : L(\lambda)) \\
&= d^2 \textstyle\sum_{\eta \in \mathsf{Irr}(K)} [P(\mu) : \Delta(\eta)](\Delta(\eta) : L(\lambda)) \\
&= d^2 (P(\mu) : L(\lambda)) \\
&= d^2 \dim \operatorname{Hom}_{\bar{H}_{\mathbf{c}}(K)}(P(\lambda), P(\mu)).
\end{aligned}$$

□

6.3 Automorphisms of rational Cherednik algebras

The group W is a finite subgroup of $GL(\mathfrak{h})$. Let us choose an element $\sigma \in N_{GL(\mathfrak{h})}(W) \subset GL(\mathfrak{h})$. Then σ is an automorphism of W and we can regard it as an algebra automorphism of $\mathbb{C}W$ by making σ act trivially on $\mathbb{C}$. Moreover σ acts naturally on $\mathfrak{h}^*$ as $(\sigma \cdot x)(y) = x(\sigma^{-1} \cdot y)$ for $x \in \mathfrak{h}^*, y \in \mathfrak{h}$, hence also on $\mathbb{C}[\mathfrak{h}^*]$ and $\mathbb{C}[\mathfrak{h}]$. Let us explicitly write $\mathcal{S}(W) = \{C_1, \dots, C_k\}$ for the set of conjugacy classes of reflections in W. Then σ permutes the C_i's and regarding σ as an element of the symmetric group S_k we write $\sigma \cdot C_i = C_{\sigma(i)}$. It can be checked from the defining relations (2.2) that the maps

$$x \mapsto \sigma(x), \qquad y \mapsto \sigma(y), \qquad w \mapsto \sigma(w), \qquad x \in \mathfrak{h}^*, y \in \mathfrak{h}, w \in W$$

define an algebra isomorphism

$$\sigma : H_{t,\mathbf{c}}(W) \xrightarrow{\sim} H_{t,\sigma(\mathbf{c})}(W),$$

where $\sigma(\mathbf{c}) = \sigma(c_1, \dots, c_k) = (c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(k)})$. Since σ normalizes W, there is a well defined action of σ on $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]^W$. Hence σ descends to an isomorphism $\sigma : \bar{H}_{\mathbf{c}}(W) \xrightarrow{\sim} \bar{H}_{\sigma(\mathbf{c})}(W)$.

Now let us consider K. By definition $W \subset N_{GL(\mathfrak{h})}(K)$, therefore elements of W act as isomorphisms between the various rational Cherednik algebras associated to K. Moreover, if we once again make the assumption that the parameter $\mathbf{c}$ is W-equivariant then we see that the elements of W actually define automorphisms of $H_{t,\mathbf{c}}(K)$. These induce automorphisms of $\bar{H}_{\mathbf{c}}(K)$ and $\widetilde{H}_{\mathbf{c}}(K)$. Let M be a module for one of the three algebras $\mathbb{C}K, \bar{H}_{\mathbf{c}}(K)$ or $\widetilde{H}_{\mathbf{c}}(K)$. Then ${}^\sigma M$ is also a module for that algebra, where $M = {}^\sigma M$ as vector spaces and if a is an element of the algebra, $m \in M$, then $a \cdot_\sigma m = \sigma^{-1}(a) \cdot m$. The following lemma is standard.

Lemma 6.10. *Let λ be a K-module, $\sigma \in W$, M a $\bar{H}_{\mathbf{c}}(K)$-module and $\tilde{M}$ a $\widetilde{H}_{\mathbf{c}}(K)$-module, then:*

1. ${}^\sigma L(\lambda) \cong L({}^\sigma\lambda)$ *and* ${}^\sigma \tilde{L}(\lambda) \cong \tilde{L}({}^\sigma\lambda)$.
2. ${}^\sigma \Delta(\lambda) \cong \Delta({}^\sigma\lambda)$ *and* ${}^\sigma \tilde{\Delta}(\lambda) \cong \tilde{\Delta}({}^\sigma\lambda)$.
3. ${}^\sigma P(\lambda) \cong P({}^\sigma\lambda)$ *and* ${}^\sigma \tilde{P}(\lambda) \cong \tilde{P}({}^\sigma\lambda)$.
4. $(M : L(\lambda)) = ({}^\sigma M : {}^\sigma\lambda)$ *and* $(\tilde{M} : \tilde{L}(\lambda)) = ({}^\sigma \tilde{M} : {}^\sigma \tilde{L}(\lambda))$.

6.4 Clifford theory

We now define an action of C_d on $\widetilde{H}_{\mathbf{c}}(K)$. For $\eta \in C_d$, choose a lift σ of η in W and let $\lambda \in \mathsf{Irr}(K)$. Define

$$\eta \cdot \lambda = {}^\sigma\lambda, \quad \eta \cdot \tilde{L}(\lambda) = {}^\sigma \tilde{L}(\lambda).$$

Note that the action of C_d is only well-defined up to isomorphism, C_d acts on the isomorphism classes of the objects in $\widetilde{H}_{\mathbf{c}}(K)$-mod. Given $\mu \in \mathsf{Irr}(K)$, the stabilizer subgroup of C_d with respect to μ will be denoted C_μ. Let $C_d^\vee = \mathrm{Hom}_{\mathrm{gp}}(C_d, \mathbb{C}^*)$ be the group of characters of C_d. There is an action of $C_d^\vee$ on the isomorphism classes of the objects in $\bar{H}_{\mathbf{c}}(W)$-mod. First let us define an action of $C_d^\vee$ on $\mathsf{Irr}(W)$:

$\delta \cdot \lambda = \lambda \otimes \delta$, for $\delta \in C_d^\vee$ and $\lambda \in \mathsf{Irr}(W)$. The stabilizer subgroup of $C_d^\vee$ with respect to λ will be denoted $C_\lambda^\vee$. Let us choose coset representatives $w_1, \dots, w_d$ of C_d in W so that $\bar{H}_{\mathbf{c}}(W) = \bigoplus_i \widetilde{H}_{\mathbf{c}}(K) \cdot w_i$. Given a $\bar{H}_{\mathbf{c}}(W)$-module M we define $\delta \cdot M = M \otimes \delta$ with action

$$hw_i \cdot (m \otimes \delta) = \delta(Kw_i)(hw_i \cdot m) \otimes \delta.$$

This action does not depend on the choice of coset representatives and one can define δ as a functor on $\bar{H}_{\mathbf{c}}(W)$-mod, though we will not require this level of generality.

Let Res_K^W and Ind_K^W be the induction and restriction functors $\mathbb{C}K\text{-mod} \leftrightarrows \mathbb{C}W\text{-mod}$. Then Clifford's Theorem allows one to compare $\mathbb{C}K$ and $\mathbb{C}W$-modules via the induction and restriction functors, see [29, Chapter 7] for details. When the quotient group is cyclic it is possible to deduce the following result (the proof of which can be found in [96, Proposition 6.1]).

Proposition 6.11. *Fix $\lambda \in \mathsf{Irr}(W)$ and write $\mathsf{Res}_K^W \lambda = \mu_1 \oplus \cdots \oplus \mu_k$, where each μ_i is nonzero and irreducible. Then*

1. *$C_{\mu_i} = (C_d^\vee / C_\lambda^\vee)^\vee \subset C_d$, hence $|C_{\mu_i}| \cdot |C_\lambda^\vee| = d$,*
2. *C_d acts transitively on the set $\{\mu_1, \dots, \mu_k\}$,*
3. *the μ_i are pairwise non-isomorphic,*
4. *$\mathsf{Ind}_K^W \mu_i = \bigoplus_{\delta \in C_d^\vee / C_\lambda^\vee} \delta \cdot \lambda$.*

To relate the action of C_d on $\widetilde{H}_{\mathbf{c}}(K)$-mod and $C_d^\vee$ on $\bar{H}_{\mathbf{c}}(W)$-mod let us introduce the semisimple algebras

$$A_W := \bar{H}_{\mathbf{c}}(W)/\mathrm{rad}\, \bar{H}_{\mathbf{c}}(W) \qquad \text{and} \qquad A_K := \widetilde{H}_{\mathbf{c}}(K)/\mathrm{rad}\, \widetilde{H}_{\mathbf{c}}(K).$$

Note that $A_K \subset A_W$ and there are natural induction and restriction functors, $\mathsf{Ind}_{A_K}^{A_W}$ and $\mathsf{Res}_{A_K}^{A_W}$. The functors

$$E_W : \mathbb{C}W\text{-mod} \to A_W\text{-mod}, \qquad E_W(\lambda) := A_W \otimes_{\bar{H}_{\mathbf{c}}(W)} \bar{H}_{\mathbf{c}}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes W} \lambda$$

$$E_K : \mathbb{C}K\text{-mod} \to A_K\text{-mod}, \qquad E_K(\mu) := A_K \otimes_{\widetilde{H}_{\mathbf{c}}(K)} \widetilde{H}_{\mathbf{c}}(K) \otimes_{\mathbb{C}[\mathfrak{h}^*]^{coW} \rtimes K} \mu$$

are equivalences of categories with $E_W(\lambda) = L(\lambda)$ and $E_K(\mu) = \tilde{L}(\mu)$ for $\lambda \in \mathsf{Irr}(W)$ and $\mu \in \mathsf{Irr}(K)$.

Lemma 6.12. *The following diagram commutes up to natural equivalences.*

$$\begin{tikzcd} \mathbb{C}W\text{-mod} \arrow[r, "E_W"] \arrow[d, shift right, "\mathsf{Res}_K^W"'] & A_W\text{-mod} \arrow[d, shift right, "\mathsf{Res}_{A_K}^{A_W}"'] \\ \mathbb{C}K\text{-mod} \arrow[r, "E_K"'] \arrow[u, shift right, "\mathsf{Ind}_K^W"'] & A_K\text{-mod} \arrow[u, shift right, "\mathsf{Ind}_{A_K}^{A_W}"'] \end{tikzcd} \tag{6.3}$$

Proof. Let us write $\mathsf{Irr}(W) = \{\lambda_1, \dots, \lambda_k\}$, $\mathsf{Irr}(K) = \{\mu_1, \dots, \mu_l\}$ and $a_{ij} \in \mathbb{N}$ such that $\mathsf{Res}_K^W \lambda_i = \oplus_j \mu_j^{\oplus a_{ij}}$. We begin by showing that the functors $E_W \circ \mathsf{Ind}_K^W$ and $\mathsf{Ind}_{A_K}^{A_W} \circ E_K$ are equivalent. The fact that $\mathbb{C}W = \bigoplus_i \lambda_i \otimes \lambda_i^*$ as a $\mathbb{C}W$-$\mathbb{C}W$-bimodule implies that $E_W(\mathbb{C}W) = \bigoplus_i L(\lambda_i) \otimes \lambda_i^*$ as a A_W-$\mathbb{C}W$-bimodule. Similarly, $E_K(\mathbb{C}K) = \bigoplus_j \tilde{L}(\mu_j) \otimes \mu_j^*$ as a A_K-$\mathbb{C}K$-bimodule. Frobenius reciprocity implies that

$$E_W \circ \mathsf{Ind}_K^W \mathbb{C}K \simeq \bigoplus_{ij} L(\lambda_i) \otimes (\mu_j^*)^{\oplus a_{ij}}$$

as a A_W-$\mathbb{C}K$-bimodule. The isomorphism $\bar{H}_{\mathbf{c}}(W) \otimes_{\bar{H}_{\mathbf{c}}(K)} \tilde{\Delta}(\mu_j) \simeq \Delta(\mathsf{Ind}_K^W \mu_j)$ implies that

$$\mathsf{Ind}_{A_K}^{A_W} \tilde{L}(\mu_j) \simeq \bigoplus_i L(\lambda_i)^{\oplus a_{ij}},$$

and thus

$$\mathsf{Ind}_{A_K}^{A_W} \circ E_K(\mathbb{C}K) \simeq \bigoplus_{ij} L(\lambda_i) \otimes (\mu_j^*)^{\oplus a_{ij}}$$

as a A_W-$\mathbb{C}K$-bimodule. Since the functors $E_W \circ \mathsf{Ind}_K^W$ and $\mathsf{Ind}_{A_K}^{A_W} \circ E_K$ are exact, Watts' Theorem ([85, Theorem 5.45]) says that $E_W \circ \mathsf{Ind}_K^W$ is naturally isomorphic to $E_W \circ \mathsf{Ind}_K^W(\mathbb{C}K) \otimes_{\mathbb{C}K} -$ and $\mathsf{Ind}_{A_K}^{A_W} \circ E_K$ is naturally isomorphic to $\mathsf{Ind}_{A_K}^{A_W} \circ E_K(\mathbb{C}K) \otimes_{\mathbb{C}K} -$. The required equivalence now follows from the general fact that if A_1 and A_2 are algebras, B, C isomorphic A_1-A_2-bimodules then fixing an isomorphism $B \to C$ defines an equivalence

$$B \otimes_{A_2} - \xrightarrow{\sim} C \otimes_{A_2} - \, : \, A_1\text{-mod} \longrightarrow A_2\text{-mod}.$$

The fact that the functors $E_K \circ \mathsf{Res}_K^W$ and $\mathsf{Res}_{A_K}^{A_W} \circ E_W$ are equivalent follows from the facts that $E_W \circ \mathsf{Ind}_K^W$ and $\mathsf{Ind}_{A_K}^{A_W} \circ E_K$ are equivalent, $(\mathsf{Ind}_K^W, \mathsf{Res}_K^W)$ and $(\mathsf{Ind}_{A_K}^{A_W}, \mathsf{Res}_{A_K}^{A_W})$ are pairs of adjoint functors and that E_K and E_W are equivalences of categories. $\square$

The functors E_W and E_K behave well with respect to the groups $C_d^\vee$ and C_d. More precisely:

Lemma 6.13. *Let $\delta \in C_d^\vee$, $g \in C_d$, $\lambda \in \mathbb{C}W$-mod and $\mu \in \mathbb{C}K$-mod, then*

$$E_W(\delta \cdot \lambda) \simeq \delta \cdot E_W(\lambda) \quad \text{and} \quad E_K(g \cdot \mu) \simeq g \cdot E_K(\mu).$$

Proof. We prove that $E_W(\delta \cdot \lambda) = \delta \cdot E_W(\lambda)$, the argument for K being similar. Consider the space $1 \otimes \lambda \otimes \delta \subset \delta \cdot \Delta(\lambda)$. For $\mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]^{coW} \subset \bar{H}_{\mathbf{c}}(W)$ we have $\mathfrak{h} \cdot (1 \otimes \lambda \otimes \delta) = 0$, thus there is a non-zero map $\Delta(\delta \cdot \lambda) \to \delta \cdot \Delta(\lambda)$. The space $1 \otimes \lambda \otimes \delta$ generates $\delta \cdot \Delta(\lambda)$ therefore the map is an isomorphism. The head of $\Delta(\delta \cdot \lambda)$ is $E_W(\delta \cdot \lambda)$ and the head of $\delta \cdot \Delta(\lambda)$ is $\delta \cdot E_W(\lambda)$. This proves the result. $\square$

Combining Proposition 6.11, the commutativity of diagram (6.3) and Lemma 6.13 we can conclude that

Proposition 6.14. *Fix $\lambda \in \mathsf{Irr}(W)$ and write $\mathsf{Res}_{A_K}^{A_W} L(\lambda) = \tilde{L}(\mu_1) \oplus \cdots \oplus \tilde{L}(\mu_k)$, where each $\tilde{L}(\mu_i)$ is nonzero, irreducible. Then*

1. *$C_{\tilde{L}(\mu_i)} = C_{\mu_i}$ and $C_{L(\lambda)}^\vee = C_\lambda^\vee$.*
2. *$C_{\tilde{L}(\mu_i)} = (C_d^\vee / C_{L(\lambda)}^\vee)^\vee \subset C_d$, hence $|C_{\tilde{L}(\mu_i)}| \cdot |C_{L(\lambda)}^\vee| = d$,*
3. *C_d acts transitively on the set $\{\tilde{L}(\mu_1), \dots, \tilde{L}(\mu_k)\}$,*

4. the $\tilde{L}(\mu_i)$ are pairwise non-isomorphic,

5. $\mathsf{Ind}\,_{A_K}^{A_W} \tilde{L}(\mu_i) = \bigoplus_{\delta \in C_d^\vee / C_{L(\lambda)}^\vee} \delta \cdot L(\lambda)$.

Since $C_d^\vee$ acts on the isomorphism classes of objects in $\bar{H}_{\mathbf{c}}(W)$-mod and C_d acts on the isomorphism classes of objects in $\widetilde{H}_{\mathbf{c}}(K)$-mod, these groups also permute the blocks of the corresponding algebras. Hence there is an action of $C_d^\vee$ on the set $\mathsf{CM}_{\mathbf{c}}(W)$ and an action of the group C_d on the block partition of $\mathsf{Irr}(K)$ with respect to $\widetilde{H}_{\mathbf{c}}(K)$.

Lemma 6.15. *The action of $C_d^\vee$ on $\mathsf{CM}_{\mathbf{c}}(W)$ is trivial since each partition in $\mathsf{CM}_{\mathbf{c}}(W)$ is a union of $C_d^\vee$ orbits.*

Proof. Let δ be a generator of $C_d^\vee$. Fix B to be a block of $\bar{H}_{\mathbf{c}}(W)$ and $\lambda \in \mathsf{Irr}(W)$ such that $L(\lambda)$ is a simple module for B. Then we must show that $L(\delta \cdot \lambda)$ is also a simple module for B. Since the baby Verma modules $\Delta(\lambda)$ and $\Delta(\delta \cdot \lambda)$ are indecomposable it suffices to show that there is a nonzero map $\Delta(\delta \cdot \lambda) \to \Delta(\lambda)$. In the notation of Lemma 6.3, $\mathbb{C}[U^*]^{coC_d}$ is isomorphic to the regular representation as a C_d-module. Let $\{f_1, \ldots, f_n\}$ be the set of generators described in Lemma 6.3. Then there exist $u_1, \ldots, u_n$ with $0 \leq u_i < a_i$ such that $g := f_1^{u_1} \cdots f_n^{u_n}$ equals δ as characters of C_d. Moreover the image of g in $\mathbb{C}[\mathfrak{h}]^{coW}$ is non-zero. The polynomial g is K-invariant therefore is annihilated by $\mathfrak{h}$ in $\bar{H}_{\mathbf{c}}(W)$. Then the required map exists and is uniquely defined by $1 \otimes \delta \cdot \lambda \xrightarrow{\sim} g \otimes \lambda$. □

6.5 Twisted symmetric algebras

We shall show that $\bar{H}_{\mathbf{c}}(W)$ is an example of a twisted symmetric algebra with respect to the group C_d. We follow the exposition given in [22, Section 1] (see also [24]). Although we do not use the properties of $\bar{H}_{\mathbf{c}}(W)$ derived from the fact that it is a symmetric algebra we recall the relevant definitions for completeness. Let A be a finite dimensional $\mathbb{C}$-algebra.

Definition 6.16. A trace function on A is a linear map $t : A \to \mathbb{C}$ such that $t(ab) = t(ba)$ for all $a, b \in A$. It is called a symmetrizing form on A, and A itself is said to be a symmetric algebra, if the morphism

$$\hat{t} : A \to \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}), \qquad a \mapsto (\hat{t}(a) : b \mapsto t(ab))$$

is an isomorphism of (A, A)-bimodules.

Proposition 6.17 ([19], Corollary 3.7)**.** *The restricted rational Cherednik algebra $\bar{H}_{\mathbf{c}}(W)$ is a symmetric algebra.*

Let A be a symmetric algebra with form t and B a subalgebra of A. Then B is said to be a symmetric subalgebra of A if the restriction of t to B is a symmetric form and A is free as a left B-module.

Lemma 6.18. *The algebra $\widetilde{H}_{\mathbf{c}}(K)$ is a symmetric subalgebra of $\bar{H}_{\mathbf{c}}(W)$.*

Proof. If $w_1, \ldots, w_d$ are left coset representatives of K in W, then the PBW property (2.1) implies that $\bar{H}_{\mathbf{c}}(W)$ is a free left $\widetilde{H}_{\mathbf{c}}(K)$-module with basis $w_1, \ldots, w_d$. The fact that the restriction of t to $\widetilde{H}_{\mathbf{c}}(K)$ is symmetrizing is clear from the proof of [19, Lemma 3.5]. □

Definition 6.19. Following [22, Definition 1.10] we say that the symmetric algebra (A, t) is the twisted symmetric algebra of a finite group G over the subalgebra B if B is a symmetric subalgebra of A and there is a family of vector subspaces $\{A_g \,|g \in G\}$ of A such that the following conditions hold:

1. $A = \bigoplus_{g\in G} A_g$,

2. $A_g A_h = A_{gh}$ for all $g, h \in G$,

3. $A_1 = B$,

4. $t(A_g) = 0$ for all $g \in G$, $g \neq 1$,

5. $A_g \cap A^\times \neq \emptyset$ for all $g \in G$ (here $A^\times$ are the units of A).

Proposition 6.20. *The symmetric algebra $\bar{H}_{\mathbf{c}}(W)$ is the twisted symmetric group algebra of the group C_d over the subalgebra $\widetilde{H}_{\mathbf{c}}(K)$.*

Proof. As in Lemma 6.18, let $w_1, \dots, w_d$ be left coset representatives of K in W and assume $C_d = \{g_1, \dots, g_d\}$, such that $Kw_i = g_i$ in $W/K = C_d$. Then $\bar{H}_{\mathbf{c}}(W)_{g_i} := \widetilde{H}_{\mathbf{c}}(K) \cdot w_i$. Conditions (1), (3) and (5) are clear. Since conjugation by w_i defines an automorphism of $\widetilde{H}_{\mathbf{c}}(K)$, condition (2) is also clear. Finally condition (4) follows from the definition of Φ given in [19, (3.5)]. □

We are now in a situation where we can apply [24, Proposition 2.3.18].

Theorem 6.21. *For $\mathcal{S} \subset \mathsf{Irr}(W)$, let $\Gamma(\mathcal{S})$ be the set of all $\mu \in \mathsf{Irr}(K)$ occurring as a summand of $\mathsf{Res}\,^W_K \lambda$ for some $\lambda \in \mathcal{S}$. Let $\mathcal{P} \in \mathsf{CM}_{\mathbf{c}}(W)$. Then there exists $\mathcal{Q} \in \mathsf{CM}_{\mathbf{c}}(K)$ such that $\Gamma(\mathcal{P}) = C_d \cdot \mathcal{Q}$. This implies that there is a bijection*

$$\mathsf{CM}_{\mathbf{c}}(W) \stackrel{1:1}{\longleftrightarrow} \mathsf{CM}_{\mathbf{c}}(K)/C_d.$$

Proof. Proposition 6.2 tells us that $\{\text{blocks of } \widetilde{H}_{\mathbf{c}}(K)\} = \mathsf{CM}_{\mathbf{c}}(K)$. This identification is C_d-equivariant. Therefore it suffices to show that theorem holds but with $\mathsf{CM}_{\mathbf{c}}(K)$ replaced by $\{\text{blocks of } \widetilde{H}_{\mathbf{c}}(K)\}$. In [24] Chlouveraki makes use of the existence of a field extension of the base field of the twisted symmetric algebra A such that the extended symmetric algebra is split-semisimple. This fact is used to prove [24, Proposition 2.3.15]. Such an extension does not exist for $\bar{H}_{\mathbf{c}}(W)$ but Proposition 6.14 is our substitute result. Now [24, Proposition 2.3.18] is applicable, with $A = \bar{H}_{\mathbf{c}}(W)$ and $\bar{A} = \widetilde{H}_{\mathbf{c}}(K)$ since its proof does not explicitly rely on the existence of a "splitting field extension". This result says that the rule $C_d^\vee \cdot \mathcal{P} \mapsto \Gamma(C_d^\vee \cdot \mathcal{P})$ defines a bijection between the set of $C_d^\vee$-orbits in $\mathsf{CM}_{\mathbf{c}}(W)$ and the C_d-orbits in $\{\text{blocks of } \widetilde{H}_{\mathbf{c}}(K)\}$. However, Lemma 6.15 says that the action of $C_d^\vee$ on $\mathsf{CM}_{\mathbf{c}}(W)$ is trivial. □

Let us note a particular situation where we can give a more precise result.

Lemma 6.22. *Let $\lambda \in \mathsf{Irr}(W)$ such that $\{\lambda\} \in \mathsf{CM}_{\mathbf{c}}(W)$. Then $\mathsf{Res}\,^W_K \lambda = \oplus_{i=1}^d \mu_i$, $\mu_i \ncong \mu_j$ for $i \neq j$ and $\{\mu_i\} \in \mathsf{CM}_{\mathbf{c}}(K)$ for $1 \leq i \leq d$.*

Proof. Again, since Proposition 6.2 tells us that $\{\text{blocks of } \widetilde{H}_{\mathbf{c}}(K)\} = \mathsf{CM}_{\mathbf{c}}(K)$ it suffice to show the statement holds with $\mathsf{CM}_{\mathbf{c}}(K)$ replaced by $\{\text{blocks of } \widetilde{H}_{\mathbf{c}}(K)\}$. Proposition 6.11 tells us that $\mathsf{Res}\,^W_K \lambda = \oplus_{i=1}^e \mu_i$ for some e dividing d and $\mu_i \ncong \mu_j$ for $i \neq j$. Moreover, there exists $g \in C_d$ such that ${}^g\mu_i = \mu_j$

and hence ${}^{g}\tilde{L}(\mu_i) = \tilde{L}(\mu_j)$. In particular, $\dim \tilde{L}(\mu_i) = \dim \tilde{L}(\mu_j) = r$ for all i, j and some $r \leq |K|$. It is shown in [48, (5.3)] that $\dim L(\lambda) = |W|$ if and only if $\{\lambda\}$ is a partition of $\mathsf{CM}_{\mathbf{c}}(W)$. Proposition 6.14 says that $\mathsf{Res}_{A_K}^{A_W} L(\lambda) = \oplus_{i=1}^{e} \tilde{L}(\mu_i)$. Comparing the dimension of both sides gives

$$|W| = e \cdot r \leq d \cdot |K| = |W|.$$

Thus $e = d$ and $r = |K|$. Again, by [48, (5.3)], $\dim \tilde{L}(\mu_i) = |K|$ implies that $\{\mu_i\}$ is a block of $\widetilde{H}_{\mathbf{c}}(K)$. $\square$

6.6 The imprimitive groups $G(m, d, n)$

Recall the definition of the imprimitive complex reflection groups $G(m, d, n)$ as given in (1.5). We fix $p = m/d$ and ζ a primitive m^{th} root of unity. Let $s_{(i,j)} \in S_n$ denote the transposition swapping i and j and let ε_i^k be the matrix in $A(m, 1, n)$ which has ones all along the diagonal except in the i^{th} position where it's entry is ζ^k. The conjugacy classes of reflections in $G(m, 1, n)$ are

$$R = \{s_{(i,j)} \varepsilon_i^k \varepsilon_j^{-k} : 1 \leq i \neq j \leq n, 0 \leq k \leq m - 1\},$$

$$S_i = \{\varepsilon_j^i : 1 \leq j \leq n\}_{1 \leq i \leq m-1}.$$

The $G(m, 1, n)$-conjugacy classes of reflections in $G(m, d, n)$ are

$$R = \{s_{(i,j)} \varepsilon_i^k \varepsilon_j^{-k} : 1 \leq i \neq j \leq n, 0 \leq k \leq m - 1\},$$

$$S_{id} = \{\varepsilon_j^{id} : 1 \leq j \leq n\}_{1 \leq i \leq p-1}.$$

The following is an application of [83, Theorem 3].

Proposition 6.23. *Let $n > 2$ or $n = 2$ and d odd, then the $G(m, 1, n)$-conjugacy classes of reflections in $G(m, d, n)$ coincide with the $G(m, d, n)$-conjugacy classes of reflections $G(m, d, n)$. When $n = 2$ and d is even the $G(m, d, 2)$-conjugacy classes of reflections in $G(m, d, 2)$ are*

$$R_1 = \{s_{(1,2)} \varepsilon_i^k \varepsilon_j^{-k} : 0 \leq k \leq m - 1, k \text{ even}\}, \qquad R_2 = \{s_{(1,2)} \varepsilon_i^k \varepsilon_j^{-k} : 0 \leq k \leq m - 1, k \text{ odd}\},$$

and

$$S_{id} = \{\varepsilon_j^{id} : 1 \leq j \leq n\}_{1 \leq i \leq p-1}.$$

The group $G(m, d, n)$ is a normal subgroup of $G(m, 1, n)$ of index d and the quotient group is the cyclic group C_d. Therefore we are in the situation considered in the previous sections. If $\mathbf{c}$ is a $G(m, d, n)$-conjugate invariant function on the set of reflections of that group then, provided $n \neq 2$ or $n = 2$ and d is odd, $\mathbf{c}$ extends by zero to a $G(m, 1, n)$-conjugate invariant function on the set of reflections of $G(m, 1, n)$. If $n = 2$ and d is even, we are restricted to considering $\mathbf{c}$ such that $\mathbf{c}(R_1) = \mathbf{c}(R_2)$. The group $C_d = \langle \varepsilon_1^p \rangle$ is a cyclic subgroup of $G(m, 1, n)$ and normalizes $G(m, d, n)$. If d is co-prime to p then $G(m, 1, n) = G(m, d, n) \rtimes C_d$, an important example of this behaviour is $G(m, m, n) \triangleleft G(m, 1, n)$. In such situations there exists an algebra isomorphism

$$H_{t,\mathbf{c}}(G(m,1,n)) \cong H_{t,\mathbf{c}}(G(m,d,n)) \rtimes C_d.$$

A specific example of this is $H_{t,(c,0)}(B_n) \cong H_{t,c}(D_n) \rtimes C_2$, where B_n and D_n are the Weyl groups of type B and D respectively (they correspond to $G(2,1,n)$ and $G(2,2,n)$).

Recall from (3.3) that the set $\mathsf{Irr}(G(m,1,n))$ is parameterized by the m-multipartitions of n, $\mathcal{P}(m,n)$, and that the set $\mathsf{Irr}(G(m,d,n))$ is parameterized by the pairs $(\{\underline{\lambda}\},\epsilon)$, where $\{\underline{\lambda}\}$ is the orbit $C_d^\vee \cdot \underline{\lambda}$ and $\epsilon \in C_{\underline{\lambda}}^\vee$. The actions of the groups C_d and $C_d^\vee = \langle \delta \rangle$ on the sets $\mathsf{Irr}(G(m,d,n))$ and $\mathsf{Irr}(G(m,1,n))$ respectively, as defined in (6.4), are then described combinatorially by the equations (3.4) and (3.3) respectively. Let us recall these actions:

$$\delta \cdot (\lambda^0,\dots,\lambda^{m-1}) = (\lambda^{m-p},\lambda^{m+1-p},\dots,\lambda^{m-2},\lambda^{m-1},\lambda^0,\lambda^1,\dots,\lambda^{m-p-1}),$$

$$\eta \cdot (\{\underline{\lambda}\},\epsilon) = (\{\underline{\lambda}\},\eta\cdot\epsilon) \quad \text{where} \quad (\eta\cdot\epsilon)(\nu) = \epsilon(\eta\nu), \quad \text{for } \eta,\nu \in C_d.$$

6.7 Residues

Recall from section 3.4 the definition of a Young diagram and its content. Given a partition λ, we define the *residue* of λ to be the Laurent polynomial in $\mathbb{Z}[x^{\pm 1}]$ given by

$$\mathrm{Res}_\lambda(x) := \sum_{(a,b)\in Y(\lambda)} x^{\mathrm{cont}(a,b)}.$$

For $r \in \mathbb{Z}$, the *r-shifted* residue of λ is defined to be $\mathrm{Res}_\lambda^r(x) := x^r \mathrm{Res}_\lambda(x)$. Let $\underline{\lambda} \in \mathcal{P}(m,n)$ and fix $\mathbf{r} \in \mathbb{Z}^m$. Then the **r**-*shifted residue* of $\underline{\lambda}$ is defined to be

$$\mathrm{Res}_{\underline{\lambda}}^{\mathbf{r}}(x) := \sum_{i=0}^{m-1} \mathrm{Res}_{\lambda^i}^{r_i}(x).$$

In order to use the combinatorics described in [51] and [73] we must change the basis of our parameter space. Recall that we have labeled the conjugacy classes of complex reflections in $G(m,1,n)$ as R and S_i. We fix $\mathbf{c}(R) = k$ and $\mathbf{c}(S_i) = c_i$. The parameters of the rational Cherednik algebra $H_{\mathbf{c}}(G(m,1,n))$ as used in [51] are $\mathbf{h} = (h, H_0, \dots, H_{m-1})$. We wish to find an expression for these parameters in terms of k and $c_1,\dots,c_{m-1}$. For the remainder of this section we make the assumption that $k \neq 0$. Without loss of generality $k = -1$. The case $k = 0$ is dealt with in section 6.9. The parameter H_0 is chosen so that $H_0 + H_1 + \cdots + H_{m-1} = 0$. Recall that ζ is a primitive m^{th} root of unity. By [49, (2.7)] we know that $h = k$ and

$$c_i = \sum_{j=0}^{m-1} \zeta^{-ij} H_j.$$

Noting that

$$\sum_{i=1}^{m-1} \zeta^{-i(r+j)} = \begin{cases} m-1 & \text{if } r+j \equiv 0 \bmod m \\ -1 & \text{otherwise,} \end{cases}$$

we have for $1 \le r \le m-1$:

$$\zeta^{-r}c_1 + \zeta^{-2r}c_2 + \cdots + \zeta^{-(m-1)r}c_{m-1} = \sum_{i=1}^{m-1} \zeta^{-ri} \sum_{j=0}^{m-1} \zeta^{-ij} H_j$$

$$= \sum_{j=0}^{m-1} H_j \sum_{i=1}^{m-1} \zeta^{-i(r+j)} = (m-1)H_{m-r} - \sum_{\substack{j=0 \\ j \neq m-r}}^{m-1} H_j = mH_{m-r}.$$

Thus for $1 \le r \le m-1$:

$$H_r = \frac{1}{m} \sum_{i=1}^{m-1} \zeta^{-i(m-r)} c_i = \frac{1}{m} \sum_{i=1}^{m-1} \zeta^{ir} c_i. \tag{6.4}$$

6.8 The Calogero-Moser partition for $C_m \wr S_n$

The results in [51] and [73] are only valid for rational values of $\mathbf{h}$. Therefore, for the remainder of this chapter, we restrict to those parameters $\mathbf{c}$ for $G(m,1,n)$ such that $\mathbf{h} = (-1, H_0, H_1, \ldots, H_{m-1}) \in \mathbb{Q}^{m+1}$. Choose $e \in \mathbb{N}$ such that $eH_i \in \mathbb{Z}$ for all $0 \le i \le m-1$ and fix

$$\mathbf{s} = (0, eH_1, eH_1 + eH_2, \ldots, eH_1 + \cdots + eH_{m-1}) \in \mathbb{Z}^m.$$

Combining [51, Theorem 2.5] with the wonderful, but difficult combinatorial result [73, Theorem 3.13] gives:

Theorem 6.24. *The multipartitions $\underline{\lambda}, \underline{\mu} \in \mathcal{P}(m,n)$ belong to the same partition of $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$ if and only if*

$$Res^{\mathbf{s}}_{\underline{\lambda}}(x^e) = Res^{\mathbf{s}}_{\underline{\mu}}(x^e).$$

The $G(m,1,n)$-conjugacy classes of $G(m,d,n)$ are R and S_{id}, where $1 \le i \le p-1$. Thus a parameter $\mathbf{c}$ for $G(m,1,n)$ is an extension by zero of a parameter for $G(m,d,n)$ if and only if $c_i = 0$ for all $i \not\equiv 0 \bmod d$. Let us therefore assume that $c_i = 0$ for $i \not\equiv 0 \bmod d$.

Lemma 6.25. *We have $c_i = 0$ for all $i \not\equiv 0 \bmod d$ if and only if $H_{i+p} = H_i$ for all i.*

Proof. First assume that $c_i = 0$ for all $i \not\equiv 0 \bmod d$. Then

$$H_{i+p} = \frac{1}{m} \sum_{r=1}^{p-1} \zeta^{dr(i+p)} c_{dr} = \frac{1}{m} \sum_{r=1}^{p-1} \zeta^{dri} c_{dr} = H_i.$$

Conversely, if $H_{i+p} = H_i$ for all i then

$$c_i = \sum_{j=0}^{m-1} \zeta^{-ij} H_j = \sum_{j=0}^{p-1} H_j \sum_{r=0}^{d-1} \zeta^{-i(j+rp)}.$$

The result now follows from

$$\sum_{r=0}^{d-1} \zeta^{-i(j+rp)} = \zeta^{-ij} \sum_{r=0}^{d-1} (\zeta^{-ip})^r = \begin{cases} d\zeta^{-ij} & \text{if } i \equiv 0 \bmod d \\ 0 & \text{otherwise.} \end{cases}$$

□

We will say that the parameter $\mathbf{h} = (-1, H_0, \dots, H_{m-1})$ is *p-cyclic* if $H_{i+p} = H_i$ for all i. Let $\underline{\lambda} = (\lambda^0, \dots, \lambda^{m-1})$ be an m-partition of n. We rewrite $\underline{\lambda}$ as $\underline{\lambda} = (\underline{\lambda}_0, \dots, \underline{\lambda}_{d-1})$ where $\underline{\lambda}_i = (\lambda^{ip}, \dots, \lambda^{(i+1)p-1})$. Now the action of $C_d^\vee$ on $\underline{\lambda}$ as defined in (3.3) can be expressed as

$$\delta \cdot (\underline{\lambda}_0, \dots, \underline{\lambda}_{d-1}) = (\underline{\lambda}_{d-1}, \underline{\lambda}_0, \dots, \underline{\lambda}_{d-2}).$$

An m-multipartition of n is called *d-stuttering* if $\underline{\lambda}_i = \underline{\lambda}_j$ for all $0 \leq i, j \leq d-1$. The group $C_d^\vee$ can be considered as a subgroup of $\mathfrak{S}_d$, the symmetric group on d elements, acting on $\mathcal{P}(m,n)$ as:

$$\sigma \cdot (\underline{\lambda}_0, \dots, \underline{\lambda}_{d-1}) = (\underline{\lambda}_{\sigma(0)}, \dots, \underline{\lambda}_{\sigma(d-1)}).$$

Lemma 6.26. *Let* $\mathbf{c}$ *be a parameter for* $G(m,1,n)$ *such that* $\mathbf{h} \in \mathbb{Q}^{m+1}$ *is p-cyclic. Then the partitions of* $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$ *consist of* $\mathfrak{S}_d$*-orbits since*

$$Res^{\mathbf{s}}_{\underline{\lambda}}(x^e) = Res^{\mathbf{s}}_{\sigma \cdot \underline{\lambda}}(x^e),$$

where $\underline{\lambda} \in \mathcal{P}(m,n)$, $\sigma \in \mathfrak{S}_d$ *and* $\mathbf{s}$ *is defined in (6.24).*

Proof. If $\mathbf{h}$ is p-cyclic then the corresponding parameter $\mathbf{s}$ has the form

$$\mathbf{s} = (\mathbf{s}', \dots, \mathbf{s}') \quad \text{where} \quad \mathbf{s}' = (0, eH_1, eH_1 + eH_2, \dots, eH_1 + \dots + eH_{p-1}),$$

and thus

$$\mathrm{Res}^{\mathbf{s}}_{\underline{\lambda}}(x^e) = \sum_{i=0}^{d-1} \mathrm{Res}^{\mathbf{s}'}_{\underline{\lambda}_i}(x^e) \qquad \forall \, \underline{\lambda} \in \mathcal{P}(m,n).$$

Since the action of $\mathfrak{S}_d$ simply reorders this sum, the result is clear. □

The following technical result will be needed later.

Lemma 6.27. *Let* $\mathbf{h}$ *be a p-cyclic parameter and choose* $\underline{\lambda} \in \mathcal{P}(m,n)$ *to be a non d-stuttering m-multipartition of n. For each prime divisor q of d, there exists an m-multipartition* $\underline{\lambda}(q)$ *of n such that* $\underline{\lambda}$ *and* $\underline{\lambda}(q)$ *belong to the same partition of* $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$ *and the order of the stabilizer of* $\underline{\lambda}(q)$ *under the action of* $C_d^\vee$ *is not divisible by q.*

Proof. We follow the argument given in [67, Lemma 3.5]. Since $\underline{\lambda}$ is not d-stuttering, there exists an $i > 0$ such that $\underline{\lambda}_i \neq \underline{\lambda}_0$. If $d = q$ there is nothing to prove so assume $d > q$ and set $l = d/q$, $l > 1$. Let σ be the transposition in $\mathfrak{S}_d$ that swaps $\underline{\lambda}_i$ and $\underline{\lambda}_{l-1}$ in $\underline{\lambda}$. We set $\underline{\lambda}(q) = \sigma \cdot \underline{\lambda}$. Then $\underline{\lambda}(q)$ is not fixed by any of the generators of the unique subgroup of $C_d^\vee$ of order q and hence the stabilizer subgroup of $\underline{\lambda}(q)$

has order co-prime to q. Since $\underline{\lambda}$ and $\underline{\lambda}(q)$ are in the same $\mathfrak{S}_d$-orbit, Lemma 6.26 says that they are in the same partition of $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$. $\square$

We will also require the following result.

Lemma 6.28. *Let* $\mathbf{c}$ *be a parameter for* $G(m,1,n)$ *such that* $\mathbf{h} \in \mathbb{Q}^{m+1}$ *is* p*-cyclic and choose* $\underline{\lambda} \in \mathcal{P}(m,n)$ *to be* d*-stuttering. If* $\{\underline{\lambda}\}$ *is not a partition of* $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$ *then there exists a non* d*-stuttering* m*-multipartition* $\underline{\mu}$ *that is in the same partition as* $\underline{\lambda}$*.*

Proof. Since $\{\underline{\lambda}\}$ is not partition of $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$ there must exist an m-multipartition $\underline{\lambda}' \neq \underline{\lambda}$ that is in the same partition as $\underline{\lambda}$. If $\underline{\lambda}'$ is not d-stuttering then we are done. Therefore we assume that $\underline{\lambda}'$ is d-stuttering. As $\mathbf{h}$ is p-cyclic, the corresponding parameter $\mathbf{s}$ has the form

$$\mathbf{s} = (\mathbf{s}', \dots, \mathbf{s}') \qquad \text{where} \qquad \mathbf{s}' = (0, eH_1, eH_1 + eH_2, \dots, eH_1 + \cdots + eH_{p-1}),$$

and thus

$$\mathrm{Res}^{\mathbf{s}}_{\underline{\mu}}(x^e) = \sum_{i=0}^{d-1} \mathrm{Res}^{\mathbf{s}'}_{\underline{\mu}_i}(x^e) \qquad \forall\, \underline{\mu} \in \mathcal{P}(m,n).$$

Hence $\mathrm{Res}^{\mathbf{s}}_{\underline{\lambda}}(x^e) = d\,\mathrm{Res}^{\mathbf{s}'}_{\underline{\lambda}_0}(x^e)$ and $\mathrm{Res}^{\mathbf{s}}_{\underline{\lambda}'}(x^e) = d\,\mathrm{Res}^{\mathbf{s}'}_{(\underline{\lambda}')_0}(x^e)$. It follows from Theorem 6.24 that

$$\mathrm{Res}^{\mathbf{s}'}_{\underline{\lambda}_0}(x^e) = \mathrm{Res}^{\mathbf{s}'}_{(\underline{\lambda}')_0}(x^e).$$

Set $\underline{\mu} = (\underline{\lambda}_0, (\underline{\lambda}')_0, \underline{\lambda}_0, \dots, \underline{\lambda}_0)$; it is a non d-stuttering m-multipartition. Again by Theorem 6.24, $\mathrm{Res}^{\mathbf{s}}_{\underline{\lambda}}(x^e) = \mathrm{Res}^{\mathbf{s}}_{\underline{\mu}}(x^e)$ implies that $\underline{\lambda}$ and $\underline{\mu}$ belong to the same partition of $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$. $\square$

Recall that for $\mathcal{P} \in \mathsf{CM}_{\mathbf{c}}(W)$, $\Gamma(\mathcal{P})$ was defined to be the set of all $\mu \in \mathsf{Irr}(K)$ occurring as a summand of $\mathsf{Res}^K_W \lambda$ for each $\lambda \in \mathcal{P}$. In the case $W = G(m,1,n)$ and $K = G(m,d,n)$, Γ is given combinatorially by $\Gamma(\mathcal{P}) = \{ (\{\underline{\lambda}\}, \epsilon) \mid \underline{\lambda} \in \mathcal{P},\ \epsilon \in C^{\vee}_{\underline{\lambda}} \}$.

Theorem 6.29. *Let* $\mathbf{c} : \mathcal{S}(G(m,d,n)) \to \mathbb{C}$ *be a* $G(m,1,n)$*-equivariant function such that* $k \neq 0$ *and* $\mathbf{h} \in \mathbb{Q}^{m+1}$*. The* $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ *partition of* $\mathsf{Irr}(G(m,d,n))$ *is described as follows. Let* $\mathcal{Q}$ *be a partition in* $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$*:*

1. *If* $\underline{\lambda}$ *is a* d*-stuttering* m*-multipartition such that* $\mathcal{Q} = \{\underline{\lambda}\}$ *then the sets* $\{(\{\underline{\lambda}\}, \epsilon)\}$ *where* $\epsilon \in C^{\vee}_d$ *are partitions of* $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$*;*

2. *Otherwise* $\Gamma(\mathcal{Q})$ *is a* $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ *partition of* $\mathsf{Irr}(G(m,d,n))$*.*

Proof. Rescaling if necessary, we may assume that $k = -1$. It is clear that the sets described in (1) and (2) of the theorem define a partition of the set $\mathsf{Irr}\,(G(m,d,n))$. Therefore we just have to show that the sets describe the blocks of $\bar{H}_{\mathbf{c}}(G(m,d,n))$. Proposition 6.2 says that it is sufficient to prove that (1) and (2) describe the equivalence classes of $\mathsf{Irr}\,(G(m,d,n))$ with respect to the blocks of $\tilde{H}_{\mathbf{c}}(G(m,d,n))$. Lemma 6.22 shows that the sets described in (1) are indeed blocks of $\tilde{H}_{\mathbf{c}}(G(m,d,n))$. So let us assume that $\mathcal{Q}$ is not of the form described in (1). The group C_d acts on the set $\Gamma(\mathcal{Q})$ and Theorem 6.21 says that there exists a block B of $\tilde{H}_{\mathbf{c}}(G(m,d,n))$ such that $C_d \cdot B = \Gamma(\mathcal{Q})$. We wish to show that $C_d \cdot B = B$. The fact that $g \cdot \tilde{L} \in g \cdot B$ for $\tilde{L} \in B$ and $g \in C_d$ implies that

$$\bigcup_{\tilde{L} \in B} \mathrm{Stab}_{C_d} \tilde{L} \subseteq \mathrm{Stab}_{C_d} B.$$

To show that $\mathrm{Stab}_{C_d} B = C_d$ we will show that for every prime q dividing d there exists a $\tilde{L} \in B$ such that the highest power of q dividing d also divides $|\mathrm{Stab}_{C_d} \tilde{L}(\mu)|$. This will imply $C_d \cdot B = B$ i.e. $\Gamma(\mathcal{Q}) = B$. Let $L(\lambda) \in \mathcal{Q}$ and let $\tilde{L}(\mu)$ be a summand of $\mathsf{Res}^{A_{G(m,1,n)}}_{A_{G(m,d,n)}} L(\lambda)$, then $\tilde{L}(\mu) \in g \cdot B$ for some $g \in C_d$. This means that $g^{-1} \cdot \tilde{L}(\mu) \in B$ is also a summand of $L(\lambda)$. Thus $\mathsf{Res}^{A_{G(m,1,n)}}_{A_{G(m,d,n)}} L(\lambda)$ contains a summand that lives in B, for all $L(\lambda) \in \mathcal{Q}$. Since $\mathrm{Stab}_{C_d} \tilde{L}(\mu) = \mathrm{Stab}_{C_d} \tilde{L}(\mu')$ for any two summands $\tilde{L}(\mu)$ and $\tilde{L}(\mu')$ of $\mathsf{Res}^{A_{G(m,1,n)}}_{A_{G(m,d,n)}} L(\lambda)$, it will suffice to show that, for every prime q dividing d, there exists a $L(\lambda) \in \mathcal{Q}$ such that the highest power of q dividing d also divides $|\mathrm{Stab}_{C_d} \tilde{L}(\mu)|$ for some summand $\tilde{L}(\mu)$ of $\mathsf{Res}^{A_{G(m,1,n)}}_{A_{G(m,d,n)}} L(\lambda)$. Proposition 6.14 (1) says that

$$|\mathrm{Stab}_{C_d} \tilde{L}(\mu)| \cdot |\mathrm{Stab}_{C_d^\vee} L(\lambda)| = d.$$

Therefore it suffices to show that we can find $L(\lambda) \in \mathcal{Q}$ such that q does not divide $|\mathrm{Stab}_{C_d^\vee} L(\lambda)|$. Since $\mathcal{Q} \neq \{\underline{\lambda}\}$ for some d-stuttering multipartition $\underline{\lambda}$, Lemma 6.28 says that there exists a non d-stuttering multipartition in $\mathcal{Q}$. Lemma 6.27 now says that the module $L(\lambda)$ we require exists in $\mathcal{Q}$. ☐

Corollary 6.30. *Let $\mathbf{c} : \mathcal{S}(G(m,d,n)) \to \mathbb{C}$ be a $G(m,1,n)$-equivariant function such that $k = -1$ and $\mathbf{h} \in \mathbb{Q}^{m+1}$, extended to a function $\mathbf{c} : \mathcal{S}(G(m,1,n)) \to \mathbb{C}$ and define $\mathbf{s}$ as in (6.24). Choose $(\{\underline{\lambda}\},\epsilon), (\{\underline{\mu}\},\eta) \in \mathsf{Irr}(G(m,d,n))$, then*

- *if $\{\underline{\lambda}\} \neq \{\underline{\mu}\}$, then $(\{\underline{\lambda}\},\epsilon)$ and $(\{\underline{\mu}\},\eta)$ are in the same partition of $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ if and only if*

$$Res^{\mathbf{s}}_{\underline{\lambda}}(x^e) = Res^{\mathbf{s}}_{\underline{\mu}}(x^e);$$

- *if $\underline{\lambda} = \underline{\mu}$ is a d-stuttering partition and $Res^{\mathbf{s}}_{\underline{\lambda}}(x^e) \neq Res^{\mathbf{s}}_{\underline{\nu}}(x^e)$ for all $\underline{\lambda} \neq \underline{\nu} \in \mathcal{P}(m,n)$ then $(\{\underline{\lambda}\},\epsilon)$ and $(\{\underline{\lambda}\},\eta)$ are in the same partition of $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ if and only if $\epsilon = \eta$;*
- *otherwise $(\{\underline{\lambda}\},\epsilon)$ and $(\{\underline{\lambda}\},\eta)$ are in the same partition of $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$.*

It was shown by the author in [7] that the partition $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ is never trivial, even for generic values of $\mathbf{c}$. Here we describe $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ for generic $\mathbf{c}$.

Lemma 6.31. *Let $\mathbf{c}$ be a generic parameter for $H_{\mathbf{c}}(G(m,d,n))$ such that $k \neq 0$ and $\mathbf{h} \in \mathbb{Q}^{m+1}$. Choose $(\{\underline{\lambda}\},\epsilon), (\{\underline{\mu}\},\eta) \in \mathsf{Irr}(G(m,d,n))$,*

- *if $\underline{\lambda}$ is a d-stuttering partition then $\{\,(\{\underline{\lambda}\},\epsilon)\,\}$ is a partition of $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$.*
- *otherwise $(\{\underline{\lambda}\},\epsilon)$ and $(\{\underline{\mu}\},\eta)$ are in the same partition of $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ if and only if*

$$\sum_{i=0}^{d-1} Res_{\lambda^{j+pi}}(x^e) = \sum_{i=0}^{d-1} Res_{\mu^{j+pi}}(x^e) \quad \forall\, 0 \leq j \leq p-1. \tag{6.5}$$

Note that the expressions in (6.5) are independent of the choice of representative $\underline{\lambda} \in \{\underline{\lambda}\}$ and $\underline{\mu} \in \{\underline{\mu}\}$.

Proof. Since $\mathbf{h}$ is cyclic, we note once again that the vector $\mathbf{s}$ as defined in (6.24) has the form

$$\mathbf{s} = (\mathbf{s}', \dots, \mathbf{s}') \quad \text{where} \quad \mathbf{s}' = (0, eH_1, eH_1 + eH_2, \dots, eH_1 + \dots + eH_{p-1}).$$

Therefore

$$\mathrm{Res}^{\mathbf{s}}{}_{\underline{\lambda}}(x^e) = \sum_{j=0}^{p-1} x^{e\mathbf{s}_j} \left(\sum_{i=0}^{d-1} \mathrm{Res}_{\lambda^{j+pi}}(x^e) \right),$$

and thus the genericity of $\mathbf{c}$ implies that

$$\mathrm{Res}^{\mathbf{s}}{}_{\underline{\lambda}}(x^e) = \mathrm{Res}^{\mathbf{s}}{}_{\underline{\mu}}(x^e) \Leftrightarrow \sum_{i=0}^{d-1} \mathrm{Res}_{\lambda^{j+pi}}(x^e) = \sum_{i=0}^{d-1} \mathrm{Res}_{\mu^{j+pi}}(x^e) \quad \forall\, 0 \le j \le p-1.$$

If $\underline{\lambda}$ is d-stuttering then $\sum_{i=0}^{d-1} \mathrm{Res}_{\lambda^{j+pi}}(x^e) = d\,\mathrm{Res}_{\lambda^j}(x^e)$, $\forall\, 0 \le j \le p-1$. Note that if λ is a partition of n then one can recover λ from knowing the polynomial $\mathrm{Res}_\lambda(x^e)$. Also, if ax^{eb}, $a \neq 0$, is the monomial of smallest degree occurring in $\mathrm{Res}_\lambda(x^e)$ then $a = 1$ (since b is the content of the highest box in the first column of $Y(\lambda)$). I claim that if

$$d\,\mathrm{Res}_{\lambda^j}(x^e) = \sum_{i=0}^{d-1} \mathrm{Res}_{\mu^{j+pi}}(x^e) \tag{6.6}$$

then $\mu^{j+pi} = \lambda^j$ for all i. This can be shown by induction on r, where $\lambda^j \vdash r$. If $r = 1$ then $\mu^{j+pi} = \lambda^j = (1)$ for all i. Therefore assume $r > 1$ and let dax^{eb}, $a \neq 0$, be the monomial of smallest degree occurring in $d\,\mathrm{Res}_{\lambda^j}(x^e)$. As noted above, a must equal one. For each i, the partition μ^{j+pi} has at most one box with content b and hence the coefficient of x^{eb} in $\mathrm{Res}_{\mu^{j+pi}}(x^e)$ is either one or zero. Then equation (6.6) implies that it must actually be one and the first column of $Y(\mu^{j+pi})$ equals the first column of $Y(\lambda^j)$. We can remove this first column from λ^j and all partitions μ^{j+pi} and conclude by induction that the claim holds. Therefore each d-stuttering partition forms a singleton partition in $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$. Now the Lemma follows from Corollary 6.29. $\square$

6.9 The case $k = 0$

In this section we deal with the case $k = 0$. The results of Gordon and Martino are not applicable here so we must first calculate the Calogero-Moser partition for $G(m,1,n)$. We exploit the fact that C_m^n is a normal subgroup of $C_m \wr S_n$ and, when $\mathbf{c} = (k, c_1, \dots, c_{m-1}) = (0, c_1, \dots, c_{m-1})$, there is an embedding of rational Cherednik algebras $H_{t,\mathbf{c}}(C_m^n) \hookrightarrow H_{t,\mathbf{c}}(C_m \wr S_n)$. Since $(\mathbb{C}^{2n}, \omega, C_m^n)$ decomposes as a direct sum of indecomposable triples $(\mathbb{C}^2, \omega', C_m)^{\oplus n}$ and $C_m \wr S_n = C_m^n \rtimes S_n$, we have

$$H_{t,\mathbf{c}}(C_m^n) = H_{t,\mathbf{c}}(C_m) \otimes \ldots \otimes H_{t,\mathbf{c}}(C_m) \hookrightarrow (H_{t,\mathbf{c}}(C_m) \otimes \ldots \otimes H_{t,\mathbf{c}}(C_m)) \rtimes S_n = H_{t,\mathbf{c}}(C_m \wr S_n), \tag{6.7}$$

with S_n acting by permuting the tensorands. This implies that $Z_{\mathbf{c}}(C_m \wr S_n) = (Z_{\mathbf{c}}(C_m)^{\otimes n})^{S_n}$ and the inclusion $Z_{\mathbf{c}}(C_m \wr S_n) \hookrightarrow Z_{\mathbf{c}}(C_m)^{\otimes n}$ corresponds to the symmetrizing map $X_{\mathbf{c}}(C_m) \times \dots \times X_{\mathbf{c}}(C_m) \twoheadrightarrow$

$S^n(X_{\mathbf{c}}(C_m))$. This fits into the commutative diagram

$$\begin{tikzcd}
X_{\mathbf{c}}(C_m) \times \cdots \times X_{\mathbf{c}}(C_m) \arrow[rr, twoheadrightarrow] \arrow[dr, twoheadrightarrow] & & S^n(X_{\mathbf{c}}(C_m)) \arrow[dl, twoheadrightarrow] \\
& (\mathbb{C}^n/C_m \wr S_n) \times ((\mathbb{C}^n)^*/C_m \wr S_n) &
\end{tikzcd}$$

The commutativity of the diagram is simply the fact that all maps come from the inclusions

$$A := \mathbb{C}[\mathbb{C}^n]^{(C_m \wr S_n)} \otimes \mathbb{C}[(\mathbb{C}^n)^*]^{(C_m \wr S_n)} \subset Z_{\mathbf{c}}(C_m \wr S_n) \subset Z_{\mathbf{c}}(C_m)^{\otimes n}.$$

Let A_+ be the ideal in A of functions with zero constant term. The Reynolds operator with respect to S_n,

$$R_{S_n} : Z_{\mathbf{c}}(C_m)^{\otimes n} \twoheadrightarrow Z_{\mathbf{c}}(C_m \wr S_n),$$

is $Z_{\mathbf{c}}(C_m \wr S_n)$-linear. Therefore $Z_{\mathbf{c}}(C_m \wr S_n) \cap (Z_{\mathbf{c}}(C_m)^{\otimes n} \cdot A_+) = Z_{\mathbf{c}}(C_m \wr S_n) \cdot A_+$ and

$$B_1 := Z_{\mathbf{c}}(C_m \wr S_n)/\langle A_+ \rangle \hookrightarrow Z_{\mathbf{c}}(C_m)^{\otimes n}/\langle A_+ \rangle =: B_2.$$

The symmetric group acts on the set of primitive idempotents in B_2, $e \mapsto g \cdot e$.

Lemma 6.32. *The rule $e \mapsto |S_n \cdot e| \cdot R_{S_n}(e) =: \tilde{e}$ defines a bijection between the set of S_n-orbits in the set of primitive idempotents of B_2 and the set of primitive idempotents in B_1.*

Proof. Note that $\tilde{e} \in B_1$. A direct calculation using the fact that

$$|S_n \cdot e| \cdot R_{S_n}(e) = \frac{1}{|\mathrm{Stab}_{S_n}(e)|} \sum_{g \in S_n} g \cdot e \tag{6.8}$$

shows that $\tilde{e}$ is an idempotent. If B is any finite dimensional, commutative $\mathbb{C}$-algebra then there is a natural bijection between the set of primitive idempotents of B and $\mathrm{Maxspec}(B)$, where the primitive idempotent e corresponds to the unique maximal ideal $\mathfrak{m} \in \mathrm{Maxspec}(B)$ such that $e \notin \mathfrak{m}$. Fix e a primitive idempotent in B_2 and $\mathfrak{m}$ the unique maximal ideal of B_2 such that $e \notin \mathfrak{m}$. If $\tilde{e} \in \bigcap_{g \in S_n} g \cdot \mathfrak{m}$ then $e \cdot \tilde{e} = e \in \bigcap_{g \in S_n} g \cdot \mathfrak{m} \subset \mathfrak{m}$, which is a contradiction. Therefore

$$\mathfrak{m} \cap B_1 \subseteq \bigcap_{g \in S_n} g \cdot \mathfrak{m} \quad \text{and} \quad \tilde{e} \notin \bigcap_{g \in S_n} g \cdot \mathfrak{m}$$

imply that $\tilde{e} \notin \mathfrak{m} \cap B_1$. Note that the map $\mathrm{Maxspec}(B_2) \rightarrow \mathrm{Maxspec}(B_1)$, $\mathfrak{m} \mapsto \mathfrak{m} \cap B_1$, is surjective. Therefore we have found a collection $\{\tilde{e} \,|\, e \text{ primitive idempotent of } B_2\}$ of orthogonal idempotents in B_1 such that for each maximal ideal $\tilde{\mathfrak{m}}$ in B_1 there is at least one idempotent not in $\tilde{\mathfrak{m}}$. This means that there is exactly one idempotent not in $\tilde{\mathfrak{m}}$ and our collection is precisely the set of primitive idempotents of B_1. $\square$

In order to understand the combinatorial meaning of Lemma 6.32 we need to study modules for $C_m \wr S_n$ that are induced from C_m^n. Recall (3.3) that we have labeled the simple $C_m = \langle \varepsilon \rangle$-modules $\omega_i, 0 \leq i \leq m-1$ where $\varepsilon \cdot \omega_i = \zeta^i \omega_i$. Then the C_m^n-modules are all of the form $\omega_{(i_1,\ldots,i_n)} := \omega_{i_1} \otimes \cdots \otimes \omega_{i_n}$ with

$$\varepsilon_j^k \cdot \omega_{i_1} \otimes \cdots \otimes \omega_{i_n} = \zeta^{k \cdot i_j} \omega_{i_1} \otimes \cdots \otimes \omega_{i_n}.$$

The n-tuple $(i_1, \ldots, i_n)$ will be written $\underline{i}$. We say that $\underline{\lambda} \in \mathcal{P}(m,n)$ is of type $\underline{i} = (i_1, \ldots, i_n)$, written $\underline{\lambda} \in \text{Type}\,(\underline{i})$, if $|\lambda^j| = \#\{i_k \in (i_1, \ldots, i_n) \,|\, i_k = j\}$ for all $0 \leq j \leq m-1$. Since the elements of $\mathsf{Irr}\,(C_m^n)$ are parameterized by the n-tuples $\underline{i} \in (\mathbb{Z}_m)^n$, there is an obvious action of S_n on $\mathsf{Irr}\,(C_m^n)$. Clearly $\underline{\lambda}$ is of type $\underline{i}$ if and only if it is of type $\sigma \cdot \underline{i}$ for each $\sigma \in S_n$.

Lemma 6.33. *Let $\omega_{\underline{i}} := \omega_{(i_1,\ldots,i_n)}$ be a simple C_m^n-module. The induced $C_m \wr S_n$-module decomposes as*

$$\mathsf{Ind}_{C_m^n}^{C_m \wr S_n} \omega_{\underline{i}} = \bigoplus_{\underline{\lambda} \in \, Type\,(\underline{i})} V_{\underline{\lambda}}^{\oplus d(\underline{\lambda})},$$

where $d(\underline{\lambda}) = \dim(V_{\lambda^0}) \times \cdots \times \dim(V_{\lambda^{m-1}})$ and V_{λ^i} is the simple $S_{|\lambda^i|}$-module labeled by λ^i.

Proof. Fix $\underline{\lambda}$ an m-multipartition of n of type $\underline{i}$. Let us try to calculate

$$\text{Hom}_{C_m \wr S_n}(\mathsf{Ind}_{C_m^n}^{C_m \wr S_n} \omega_{\underline{i}}, V_{\underline{\lambda}}) = \text{Hom}_{C_m^n}(\omega_{\underline{i}}, V_{\underline{\lambda}}). \tag{6.9}$$

We choose $\sigma \in S_n$ such that $0 \leq i_{\sigma(1)} \leq i_{\sigma(2)} \leq \cdots \leq i_{\sigma(n)} \leq m-1$. Recall from (3.3) that

$$V_{\underline{\lambda}} := \mathsf{Ind}_{G_{(n)}}^{C_m \wr S_n} (\omega_0 \wr V_{\lambda^0}) \otimes \cdots \otimes (\omega_{m-1} \wr V_{\lambda^{m-1}}),$$

where $G_{(n)} := C_m \wr S_{(n)}$ and $S_{(n)} := S_{|\lambda^0|} \times \cdots \times S_{|\lambda^{m-1}|}$. Note that $G_{(n)}$ only depends, up to isomorphism, on the type of $\underline{\lambda}$. The space

$$\text{Hom}_{C_m^n}(\omega_{\underline{i}}, \mathbb{C} \cdot (\sigma\, G_{(n)}) \otimes_{G_{(n)}} (\omega_0 \wr V_{\lambda^0}) \otimes \cdots \otimes (\omega_{m-1} \wr V_{\lambda^{m-1}}))$$

is a subspace of (6.9). For our choice of σ,

$$\mathbb{C} \cdot (\sigma\, G_{(n)}) \otimes_{G_{(n)}} (\omega_0 \wr V_{\lambda^0}) \otimes \cdots \otimes (\omega_{m-1} \wr V_{\lambda^{m-1}}) \simeq \omega_{\underline{i}} \otimes (V_{\lambda^0} \otimes \cdots \otimes V_{\lambda^{m-1}})$$

as a C_m^n-module. Therefore

$$\dim \text{Hom}_{C_m^n}(\omega_{\underline{i}}, \mathbb{C} \cdot (\sigma\, G_{(n)}) \otimes_{G_{(n)}} (\omega_0 \wr V_{\lambda^0}) \otimes \cdots \otimes (\omega_{m-1} \wr V_{\lambda^{m-1}})) = \dim(V_{\lambda^0}) \times \cdots \times \dim(V_{\lambda^{m-1}}) = d(\underline{\lambda})$$

and

$$\bigoplus_{\underline{\lambda} \in \text{Type}\,(\underline{i})} V_{\underline{\lambda}}^{\oplus d(\underline{\lambda})} \subseteq \mathsf{Ind}_{C_m^n}^{C_m \wr S_n} \omega_{\underline{i}}. \tag{6.10}$$

To get an equality of spaces in (6.10) we just need to calculate the dimension of both spaces. First,

$|C_m \wr S_n| = n! \cdot m^n$, $|G_{(n)}| = m^n \cdot |S_{(n)}|$ and

$$\dim V_{\underline{\lambda}} = |C_m \wr S_n / G_{(n)}| \cdot d(\underline{\lambda}) = \frac{n! \cdot d(\underline{\lambda})}{|S_{(n)}|}.$$

Therefore

$$\dim \left(\bigoplus_{\underline{\lambda} \in \mathrm{Type}\,(\underline{i})} V_{\underline{\lambda}}^{\oplus d(\underline{\lambda})} \right) = \sum_{\underline{\lambda} \in \mathrm{Type}\,(\underline{i})} d(\underline{\lambda}) \cdot \frac{n! \cdot d(\underline{\lambda})}{|S_{(n)}|} = \frac{n!}{|S_{(n)}|} \sum_{\underline{\lambda} \in \mathrm{Type}\,(\underline{i})} d(\underline{\lambda})^2$$

$$= \frac{n!}{|S_{(n)}|} \cdot |S_{(n)}| = n! = \dim \mathsf{Ind}_{C_m^n}^{C_m \wr S_n} \omega_{\underline{i}}.$$

□

Given a parameter $\mathbf{c}$ with $k = 0$, written as $(0, H_0, \dots, H_{m-1})$ as in (6.7), we define

$$\mathbf{s} = (0, mH_0, mH_0 + mH_1, \dots, mH_0 + \cdots + mH_{m-2}).$$

Note that this is **not** the same as the $\mathbf{s}$ defined in section 6.8.

Proposition 6.34. *Let $\omega_{\underline{i}}$ and $\omega_{\underline{j}}$ be elements in $\mathsf{Irr}(C_m^n)$. They are in the same Calogero-Moser partition if and only if $\mathbf{s}_{i_k} = \mathbf{s}_{j_k}$ for all $1 \le k \le n$.*

Proof. Using the description of $H_{0,\mathbf{c}}(C_m^n)$ as in (6.7) we see that

$$\bar{H}_{\mathbf{c}}(C_m^n) = \bar{H}_{\mathbf{c}}(C_m) \otimes \cdots \otimes \bar{H}_{\mathbf{c}}(C_m).$$

and the primitive central idempotents of $\bar{H}_{\mathbf{c}}(C_m^n)$ are all of the form $e_{i_1} \otimes \cdots \otimes e_{i_n}$, where e_{i_j} is a primitive central idempotent in $\bar{H}_{\mathbf{c}}(C_m)$. The Proposition is then a consequence of the calculations in (2.7). □

Theorem 6.35. *Let $k = 0$, then the elements of the Calogero-Moser partition $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$ are in bijection with the S_n-orbits in $\mathsf{CM}_{\mathbf{c}}(C_m^n)$. The bijection is given by*

$$S_n \cdot \mathcal{P} \mapsto \{ \underline{\lambda} \in \mathit{Type}\,(\underline{i}) \,|\, \underline{i} \in \mathcal{P} \},$$

where $\mathcal{P} \in \mathsf{CM}_{\mathbf{c}}(C_m^n)$.

Proof. It is a consequence of Müller's Theorem, as explained in Proposition 6.2, that the natural inclusions $B_1 \hookrightarrow \bar{H}_{\mathbf{c}}(C_m \wr S_n)$ and $B_2 \hookrightarrow \tilde{H}_{\mathbf{c}}(C_m^n)$ identify central primitive idempotents. By Proposition 6.2 the Calogero-Moser partition $\mathsf{CM}_{\mathbf{c}}(C_m^n)$ equals the partition of $\mathsf{Irr}\,(C_m^n)$ induced by the blocks of $\widetilde{H}_{\mathbf{c}}(K)$. Therefore Lemma 6.32 says that the S_n-orbits in $\mathsf{CM}_{\mathbf{c}}(C_m^n)$ are in bijection with the elements of $\mathsf{CM}_{\mathbf{c}}(C_m \wr S_n)$. It remains to give an explicit combinatorial description of this bijection. Let us fix $K := C_m^n$ and $W := C_m \wr S_n$. Recall from (6.4) the functors $E_W : \mathbb{C}W\text{-mod} \to A_W\text{-mod}$ and $E_K : \mathbb{C}K\text{-mod} \to A_K\text{-mod}$. It was shown in Lemma 6.12 that these functors commute, up to natural equivalences, with the induction and restriction functors. Therefore Lemma 6.33 implies that

$$\mathsf{Ind}_{A_K}^{A_W} \tilde{L}(\omega_{\underline{i}}) = \bigoplus_{\underline{\lambda} \in \mathrm{Type}\,(\underline{i})} L(V_{\underline{\lambda}})^{\oplus d(\underline{\lambda})}.$$

Therefore, if $\underline{\lambda}$ is of type $\underline{i}$, $L(V_{\underline{\lambda}})$ occurs with non-zero multiplicity in the head of $\bar{H}_{\mathbf{c}}(W) \otimes_{\widetilde{H}_{\mathbf{c}}(K)} \tilde{L}(\omega_{\underline{i}})$. Let $e \in B_2$ be the primitive idempotent corresponding to the block to which $\tilde{L}(\omega_{\underline{i}})$ belongs. Then the explicit expression for $\tilde{e}$ given in equation (6.8) shows that

$$\tilde{e} \cdot \tilde{L}(\omega_{\underline{i}}) = e \cdot \tilde{L}(\omega_{\underline{i}}) = \tilde{L}(\omega_{\underline{i}}).$$

Hence

$$\tilde{e} \cdot \bar{H}_{\mathbf{c}}(W) \otimes_{\widetilde{H}_{\mathbf{c}}(K)} \tilde{L}(\omega_{\underline{i}}) = \bar{H}_{\mathbf{c}}(W) \otimes_{\widetilde{H}_{\mathbf{c}}(K)} \tilde{e} \cdot \tilde{L}(\omega_{\underline{i}}) = \bar{H}_{\mathbf{c}}(W) \otimes_{\widetilde{H}_{\mathbf{c}}(K)} \tilde{L}(\omega_{\underline{i}}),$$

which implies that $\tilde{e}$ will act as the identity on $L(V_{\underline{\lambda}})$. This proves the Theorem. $\square$

Example 6.36. Let us fix $m = 4$ $n = 2$ and choose $\mathbf{c}$ such that the blocks of $\bar{H}_{\mathbf{c}}(C_4)$ are $\{0,1,2\}, \{3\}$. Then blocks of $\bar{H}_{\mathbf{c}}(C_4^2)$ are

$$A = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$$

$$B = \{(3,0),(3,1),(3,2)\},\ C = \{(0,3),(1,3),(2,3)\},\ D = \{(3,3)\}$$

The S_2-orbits are $A, B \cup C, D$ and corresponding blocks for $\bar{H}_{\mathbf{c}}(C_4 \wr S_2)$ are

$$A \leftrightarrow \{(\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset,\emptyset,\emptyset),(\square\square,\emptyset,\emptyset,\emptyset),(\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset,\emptyset),$$

$$(\emptyset,\square\square,\emptyset,\emptyset),(\emptyset,\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix},\emptyset),(\emptyset,\emptyset,\square\square,\emptyset),(\square,\square,\emptyset,\emptyset),(\square,\emptyset,\square,\emptyset),(\emptyset,\square,\square,\emptyset)\},$$

$$B \cup C \leftrightarrow \{(\square,\emptyset,\emptyset,\square),(\emptyset,\square,\emptyset,\square),(\emptyset,\emptyset,\square,\square)\},$$

$$D \leftrightarrow \{(\emptyset,\emptyset,\emptyset,\begin{smallmatrix}\square\\\square\end{smallmatrix}),(\emptyset,\emptyset,\emptyset,\square\square)\}.$$

Recall (6.29) that the map Γ from subsets of $\mathsf{Irr}\,(C_m \wr S_n)$ to subsets of $\mathsf{Irr}\,(G(m,d,n))$ is given combinatorially by $\Gamma(\mathcal{P}) = \{\,(\,\{\underline{\lambda}\}\,,\epsilon)\,|\,\underline{\lambda} \in \mathcal{P},\ \epsilon \in C_{\underline{\lambda}}^{\vee}\}$.

Theorem 6.37. *Let $k = 0$, then the Calogero-Moser partition for $G(m,d,n)$ is described as follows:*

$$\mathsf{CM}_{\mathbf{c}}(G(m,d,n)) = \{\,\Gamma(\mathcal{P})\,|\,\mathcal{P} \in \mathsf{CM}_{\mathbf{c}}(C_m \wr S_n)\,\}.$$

Proof. We repeat the argument given in the proof of Theorem 6.29, which is itself based on Theorem 6.21. It is explained in the proof of Theorem 6.29 that in order to show that $\Gamma(\mathcal{P})$ is a block for $\bar{H}_{\mathbf{c}}(G(m,d,n))$, where $\mathcal{P} \in \mathsf{CM}_{\mathbf{c}}(C_m \wr S_n)$, it suffices to show that there exists some $\underline{\lambda}$ in $\mathcal{P}$ such that $\mathrm{Stab}_{C_d^{\vee}}(\underline{\lambda}) = 1$. Lemma 6.25 shows that $c_i = 0$ for all $i \not\equiv 0 \bmod d$ implies that $\mathbf{s}_{i+p} = \mathbf{s}_i$ for all i. Therefore Proposition 6.34 says that $\underline{i}$ and $\underline{j}$ are in the same block for $\bar{H}_{\mathbf{c}}(C_m^n)$ if $i_k = j_k \bmod p$ for all $1 \le k \le n$. This means that, even though the set $\mathrm{Type}\,(\underline{i}) \subset \mathcal{P}(m,n)$ is not preserved by $C_d^{\vee}$, the set $\bigcup_{\underline{i} \in \mathcal{Q}} \mathrm{Type}\,(\underline{i})$ where $\mathcal{Q} \in \mathsf{CM}_{\mathbf{c}}(C_m^n)$ is preserved by $C_d^{\vee}$. Let us fix $\underline{i}$ to be the type of $\underline{\lambda}$ (clearly there is some choice in choosing $\underline{i}$). For each i_k we let j_k be the number $0 \le j_k \le p-1$ such that $i_k = j_k \bmod p$. Then $\underline{i}$ and $\underline{j}$ are in the same block of $\bar{H}_{\mathbf{c}}(C_m^n)$ and the m-multipartitions of n of type $\underline{j}$ all lie in the same Calogero-Moser partition $\mathcal{P}$ as $\underline{\lambda}$. If $\underline{\mu}$ is of type $\underline{j}$ then $\mu^i = \emptyset$ for all $i \ge p$. Since $\underline{\mu} \neq \emptyset$, this implies that $\mathrm{Stab}_{C_d^\vee}(\underline{\mu}) = 1$ as required. $\square$

6.10 The Calogero-Moser partition for the dihedral groups

As explained in (6.6) it is not possible to use the methods of that section to calculate the Calogero-Moser partition for certain rank two complex reflection groups. Precisely, this is true for the dihedral groups whose order is divisible by four. Here we describe the Calogero-Moser partition for the dihedral groups, partly using previous work and partly by direct calculation. Let $I_2(m) = G(m,m,2)$ denote the dihedral group of order $2m$ with $m \geq 3$. We fix presentations

$$I_2(m) = \langle a, b \,|\, a^m = b^2 = 1, bab = a^{-1} \rangle = \langle s_1, s_2 \,|\, s_1^2 = s_2^2 = (s_1 s_2)^m = 1 \rangle$$

where $s_1 = b$ and $s_2 = ab$. Recall that we have fixed ζ to be a primitive m^{th} root of unity. Since we wish to compare the representation theory of the groups $I_2(m)$ and $G(m,1,2)$ we note here that the embedding $I_2(m) \hookrightarrow G(m,1,2)$ is given by $a \mapsto \zeta_1 \zeta_2^{-1}$ and $b \mapsto s_{12}$ (in the notation of (3.3)). We study the cases m even and m odd separately. Let us begin by assuming that m is odd. The conjugacy classes of $I_2(m)$ are

$$\{1\}, \, \{a^r, a^{-r}\} \, (1 \leq r \leq (m-1)/2), \, \{a^s b : 0 \leq s \leq m-1\}.$$

There are $(m-1)/2$ non-isomorphic irreducible modules $\mathfrak{h}_1, \ldots, \mathfrak{h}_{(m-1)/2}$ of rank two, and two linear characters T, S. The character table for $I_2(m)$ is:

rep	1	$a^r (1 \leq r \leq (m-1)/2)$	b
size	1	2	m
T	1	1	1
S	1	1	-1
$\mathfrak{h}_j$	2	$\zeta^{jr} + \zeta^{-jr}$	0

The simple modules for $G(m,1,2)$ are $V_{\underline{\lambda}}$ where $\underline{\lambda}$ is one of

$$a(i) = (\emptyset, \ldots, (1,1), \ldots, \emptyset), \, b(i) = (\emptyset, \ldots, (2), \ldots, \emptyset) \text{ or } c(i,j) = (\emptyset, \ldots, (1), \ldots, (1), \ldots, \emptyset).$$

where $0 \leq i \neq j \leq m-1$. In the notation of (3.3),

$$V_{a(i)} = \omega_i \wr V_{(1,1)}, \, V_{b(i)} = \omega_i \wr V_{(2)} \text{ and } V_{c(i,j)} = \mathsf{Ind}_{C_m \times C_m}^{G(m,1,2)} (\omega_i \wr V_{(1)} \otimes \omega_j \wr V_{(1)}).$$

We have $(V_{a(i)})_{|\, I_2(m)} \simeq T, (V_{b(i)})_{|\, I_2(m)} \simeq S$. The module $V_{c(i,j)}$ has basis $\omega_i \otimes \omega_j$ and $s_{12} \cdot \omega_i \otimes \omega_j$. Then $a \cdot \omega_i \otimes \omega_j = \zeta^{i-j} \omega_i \otimes \omega_j$ and $a \cdot s_{12} \cdot \omega_i \otimes \omega_j = \zeta^{j-i} s_{12} \cdot \omega_i \otimes \omega_j$ imply that $(V_{c(i,j)})_{|\, I_2(m)} \simeq \mathfrak{h}_{j-i}$ (the index taken modulo $(m-1)/2$). In this case it is possible to deduce the Calogero-Moser partition for $I_2(m)$ from the Calogero-Moser partition for $G(m,1,2)$ using Theorem 6.29.

Proposition 6.38. *Let m be odd and $\mathbf{c} \in \mathbb{C}\backslash\{0\}$. Then the Calogero-Moser partition for $I_2(m)$ is*

$$\mathsf{CM}_{\mathbf{c}}(I_2(m)) = \{\,\{\mathfrak{h}_i \,|\, 1 \le i \le (m-1)/2\},\ \{T\},\ \{S\}\,\}.$$

Proof. We can assume without loss of generality that $\mathbf{c} = -1$. Then the corresponding parameter for $G(m,1,2)$ becomes $(h, H_1, \ldots, H_{m-1}) = (-1, 0, \ldots, 0)$ and the residue weighting is $\mathbf{s} = 0$. From the definition of residue we see that $\mathrm{Res}_{a(i)}(x) = 1 + x^{-1}$, $\mathrm{Res}_{b(i)}(x) = 1 + x$ and $\mathrm{Res}_{c(i,j)}(x) = 2$. Therefore the Calogero-Moser partition for $G(m,1,n)$ is $\{a(i) \,|\, 0 \le i \le m-1\}$, $\{b(i) \,|\, 0 \le i \le m-1\}$ and $\{c(i,j) \,|\, 0 \le i < j \le m-1\}$. Applying Theorem 6.29 gives the above result. $\square$

The situation is much more complicated when m is even because the group $I_2(m)$ contains two conjugacy classes of reflections that form a single conjugacy class in $G(m,1,2)$. For the remainder of this section we assume $m = 2n$ is even. The conjugacy classes of $I_2(m)$ are

$$\{1\},\ \{a^n\},\ \{a^r, a^{-r}\}\ (1 \le r \le n-1),\ \{a^s b : s \text{ even }\},\ \{a^s b : s \text{ odd }\}.$$

There are $n-1$ irreducible modules $\mathfrak{h}_1, \ldots, \mathfrak{h}_{n-1}$ of rank two and four linear characters T, S, V_1, V_2. The character table of $I_2(m)$ is

rep	1	a^n	$a^r (1 \le r \le n-1)$	b	ab
size	1	1	2	n	n
T	1	1	1	1	1
S	1	1	1	-1	-1
V_1	1	$(-1)^n$	$(-1)^r$	1	-1
V_2	1	$(-1)^n$	$(-1)^r$	-1	1
$\mathfrak{h}_j$	2	$2(-1)^j$	$\zeta^{jr} + \zeta^{-jr}$	0	0

(6.11)

We denote the conjugacy class of reflections containing b by C_1 and that containing ab by C_2. We write the parameter $\mathbf{c}$ as $\mathbf{c} = (c_1, c_2)$ so that $\mathbf{c}(C_1) = c_1$ and $\mathbf{c}(C_2) = c_2$. The embedding $I_2(m) \hookrightarrow G(m,1,2)$ only extends to an embedding of rational Cherednik algebras when $c_1 = c_2$. We will apply Theorem 6.29 in the case $\mathbf{c} = (1,1)$. However there are two other parameters for which we can deduce the Calogero-Moser partition using the theory developed above. When $\mathbf{c} = (1,0)$ or $\mathbf{c} = (0,1)$ it is possible to extend embeddings of $I_2(n)$ into $I_2(m)$ to algebra embedding. We will then apply Theorem 6.21 in reverse to deduce the Calogero-Moser partition from Proposition 6.38.

Lemma 6.39. *Let $m = 2n$ be even and $\mathbf{c} = (1,1)$. Then the Calogero-Moser partition for $I_2(m)$ is*

$$\mathsf{CM}_{\mathbf{c}}(I_2(m)) = \{\,\{V_1, V_2, \mathfrak{h}_i \,|\, 1 \le i \le n-1\},\ \{T\},\ \{S\}\,\}.$$

Proof. As in the case for m odd, the Calogero-Moser partition for $G(m,1,2)$ is $\{a(i) \,|\, 0 \le i \le m-1\}$, $\{b(i) \,|\, 0 \le i \le m-1\}$ and $\{c(i,j) \,|\, 0 \le i < j \le m-1\}$. Following the same arguments as before we see that $(V_{a(i)})_{|\,I_2(m)} \simeq T$, $(V_{b(i)})_{|\,I_2(m)} \simeq S$ and, provided that $j \neq i+n$, $(V_{c(i,j)})_{|\,I_2(m)} \simeq \mathfrak{h}_{j-i}$. However, when $j = i+n$, $(V_{c(i,j)})_{|\,I_2(m)} \simeq V_1 \oplus V_2$. The Lemma now follows from Theorem 6.29. $\square$

We can use Lemma 6.39 to deduce the Calogero-Moser partition for $I_2(m)$ at $\mathbf{c} = (1, -1)$.

Lemma 6.40. *Let $m = 2n$ be even and $\mathbf{c} = (1, -1)$. Then the Calogero-Moser partition for $I_2(m)$ is*

$$\mathsf{CM}_{\mathbf{c}}(I_2(m)) = \{\, \{T, S, \mathfrak{h}_i \,|\, 1 \le i \le n-1\}, \, \{V_1\}, \, \{V_2\}\, \}.$$

Proof. Let χ be the linear character of $I_2(m)$ corresponding to the one dimensional representation V_1. As explained in [44, §5.4.1], the map $\sigma : H_{0,(1,1)}(I_2(m)) \xrightarrow{\sim} H_{0,(1,-1)}(I_2(m))$ which is the identity on $\mathfrak{h}$ and $\mathfrak{h}^*$ and sends w to $\chi(w) \cdot w$, for all $w \in I_2(m)$, is an isomorphism ("twisting by the linear character χ"). We have ${}^{\sigma}L(\lambda) = L(\lambda \otimes V_1)$ for all $\lambda \in \mathsf{Irr}(I_2(m))$. The Lemma now follows from Lemma 6.39 once we note that $S \otimes V_1 = V_2$, $V_1 \otimes V_1 = T$ and $\mathfrak{h}_i \otimes V_1 = \mathfrak{h}_{n-i}$. $\square$

Let us now consider the case $\mathbf{c} = (1, 0)$. If $I_2(n) = \langle a', b' \rangle$ then we can embed $I_2(n)$ in $I_2(m)$ by $a' \mapsto a^2$ and $b' \mapsto b$ and the defining relations show that this extends to an embedding of rational Cherednik algebras $H_{0,(\mathbf{1})}(I_2(n)) \hookrightarrow H_{0,(1,0)}(I_2(m))$.

Lemma 6.41. *Let $m = 2n$ be even and $\mathbf{c} = (1, 0)$. Then the Calogero-Moser partition for $I_2(m)$ is*

$$\mathsf{CM}_{\mathbf{c}}(I_2(m)) = \{\, \{\mathfrak{h}_i \,|\, 1 \le i \le n-1\}, \, \{T, V_1\}, \, \{S, V_2\}\, \}.$$

Proof. There are two cases to consider.
Case 1
n is odd. By Proposition 6.38, the Calogero-Moser partition for $I_2(n)$ at $\mathbf{c} = (\mathbf{1})$ is $\{T\}, \{S\}, \{\mathfrak{h}_i : 1 \le i \le \frac{n-1}{2}\}$. We have $(V_1)_{| I_2(n)} \simeq T$, $(V_2)_{| I_2(n)} \simeq S$ and $(\mathfrak{h}_i)_{| I_2(n)} \simeq \mathfrak{h}_i$. Theorem 6.29 implies that the Calogero-Moser partition for $I_2(m)$ must be the one written above.
Case 2
n is even. By Lemma 6.39, the Calogero-Moser partition for $I_2(n)$ at $\mathbf{c} = (\mathbf{1}, \mathbf{1})$ is $\{T\}, \{S\}, \{V_1, V_2, \mathfrak{h}_i : 1 \le i \le \frac{n}{2} - 1\}$. We have $(V_1)_{| I_2(n)} \simeq T$, $(V_2)_{| I_2(n)} \simeq S$, $(\mathfrak{h}_i)_{| I_2(n)} \simeq \mathfrak{h}_i$, for $i \neq \frac{n}{2}$ and $(\mathfrak{h}_{\frac{n}{2}})_{| I_2(n)} \simeq V_1 \oplus V_2$. Apply Theorem 6.29 once more. $\square$

Repeating the above argument but with the embedding of $I_2(n) = \langle a', b' \rangle$ into $I_2(m)$ by $a' \mapsto a^2$ and $b' \mapsto ab$ gives:

Lemma 6.42. *Let $m = 2n$ be even and $\mathbf{c} = (0, 1)$. Then the Calogero-Moser partition for $I_2(m)$ is*

$$\mathsf{CM}_{\mathbf{c}}(I_2(m)) = \{\, \{\mathfrak{h}_i \,|\, 1 \le i \le n-1\}, \, \{T, V_2\}, \, \{S, V_1\}\, \}.$$

Now we consider the cases not covered above. We must make a direct calculation to uncover the Calogero-Moser partition in these cases (actually we'll show that there is only one other case - the "generic" situation).

Lemma 6.43. *Choose $\mathbf{c} \in \mathbb{C}^{\mathbf{2}}$ such that it does not lie on any of the four lines spanned by the vectors $(1, 0), (0, 1), (1, 1)$ and $(1, -1)$. Then the sets $\{T\}$, $\{S\}$, $\{V_1\}$ and $\{V_2\}$ are blocks in the Calogero-Moser partition $\mathsf{CM}_{\mathbf{c}}(I_2(m))$.*

Lemma 6.44. *The modules $\mathfrak{h}_1, \dots, \mathfrak{h}_{n-1}$ always belong to the same partition of $\mathsf{CM}_{\mathbf{c}}(I_2(m))$, regardless of the value of the parameter $\mathbf{c}$.*

The proof of Lemmata 6.43 and 6.44 are direct calculations and are given in the appendix, (A.3).

Proposition 6.45. *Let $m = 2n$ be even and choose $\mathbf{c} \in \mathbb{C}^{\mathbf{2}}$ such that it does not lie on one of the four lines spanned by the vectors $(1,0),(0,1),(1,1)$ and $(1,-1)$. Then the Calogero-Moser partition is*

$$\mathsf{CM}_{\mathbf{c}}(I_2(m)) = \{ \{T\}, \{S\}, \{V_1\}, \{V_2\}, \{\mathfrak{h}_{\mathrm{i}} \mid 1 \le i \le n-1\} \}.$$

Proof. Follows from Lemmata 6.43 and 6.44. □

This completes the classification of block partitions for the dihedral groups.

6.11 Remarks

1. The main results in this chapter have appear in the preprint [4].
2. In this chapter we have focused on the particular case of $W = G(m,1,n)$ and $K = G(m,d,n)$. However, we believe that it is advantageous to present Theorem 6.21 in the level of generality that we have done since there are many examples among the 34 exceptional irreducible complex reflection groups of pairs (W,K). Therefore in order to calculate the Calogero-Moser partition for all exceptional groups it would suffice to consider only certain groups. We refer the reader to the appendix of [24] for a list of many such pairs (W,K).
3. It has been pointed out to the author by Cédric Bonnafé that the methods of this chapter can also be applied to pairs (W,K) when W/K is not necessarily a cyclic group. However it does not seem likely that one can state such a general result as Theorem 6.21 in this case.

Chapter 7

Relation to Hecke algebras

The purpose of this chapter is to show that Theorem 6.29 confirms, in the case $W = G(m, d, n)$, a conjecture, made originally by Gordon and Martino [51] and refined by Martino [73], relating the Calogero-Moser partition with Rouquier families for cyclotomic Hecke algebras.

7.1 Generic Hecke algebras

To each complex reflection group it is possible to associate a generic Hecke algebra. We recall the definition as given in [73] (see also [13]). Denote by $\mathcal{K}$ the set of all hyperplanes in $\mathfrak{h}$ that are the fixed point sets of the complex reflections in W. The group W acts on $\mathcal{K}$. Given $H \in \mathcal{K}$, the parabolic subgroup of W that fixes H point-wise is a rank one complex reflection group and thus isomorphic to the cyclic group C_e for some e. Therefore an orbit of hyperplanes $\mathcal{C} \in \mathcal{K}$ corresponds to a conjugacy class of rank one parabolic subgroups, all isomorphic to $C_{e_{\mathcal{C}}}$. For every $d > 1$, fix $\eta_d = e^{\frac{2\pi i}{d}}$ and let μ_d be the group of all d^{th} roots of unity in $\mathbb{C}$. If μ_∞ is the group of all roots of unity in $\mathbb{C}$ then we choose K to be some finite field extension of $\mathbb{Q}$ contained in $\mathbb{Q}(\mu_\infty)$ such that K contains $\mu_{e_{\mathcal{C}}}$ for all $\mathcal{C} \in \mathcal{K}/W$. The group of roots of unity in K is denoted $\mu(K)$ and the ring of integers in K is $\mathbb{Z}_K$.

Fix a point $x_0 \in \mathfrak{h}_{\mathrm{reg}} := \mathfrak{h} \backslash \bigcup_{H \in \mathcal{K}} H$ and denote by $\bar{x}_0$ its image in $\mathfrak{h}_{\mathrm{reg}}/W$. Let B denote the fundamental group $\Pi_1(\mathfrak{h}_{\mathrm{reg}}/W, \bar{x}_0)$. Let $\mathbf{u} = \{ (u_{\mathcal{C},j}) \, : \mathcal{C} \in \mathcal{K}/W, \, 0 \leq j \leq e_{\mathcal{C}} - 1\}$ be a set of indeterminates, and denote by $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ the ring $\mathbb{Z}[u_{\mathcal{C},j}^{\pm 1} \, : \mathcal{C} \in \mathcal{K}/W, \, 0 \leq j \leq e_{\mathcal{C}} - 1]$. The *generic Hecke algebra*, $\mathcal{H}_W$, is the quotient of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ by the relations of the form

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1}) \cdots (\mathbf{s} - u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

where $\mathcal{C} \in \mathcal{K}/W$ and $\mathbf{s}$ runs over the set of monodromy generators around the images in $\mathfrak{h}_{\mathrm{reg}}/W$ of the hyperplane orbit $\mathcal{C}$. The following properties are known to hold for all but finitely many complex reflection groups (it is conjectured that they hold for all complex reflection groups). In particular, they hold for the infinite series $G(m, d, n)$.

- $\mathcal{H}_W$ is a free $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module of rank $|W|$.

- $\mathcal{H}_W$ has a symmetrizing form $t : \mathcal{H}_W \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ that coincides with the standard symmetrizing form on $\mathbb{Z}_K W$ after specializing $u_{\mathcal{C},j}$ to $\eta_{e_\mathcal{C}}^j$.
- Let $\mathbf{v} = \{(v_{\mathcal{C},j}) : \mathcal{C} \in \mathcal{K}/W, 0 \leq j \leq e_\mathcal{C} - 1\}$ be a set of indeterminates such that $u_{\mathcal{C},j} = \eta_{e_\mathcal{C}}^j v_{\mathcal{C},j}^{|\mu(K)|}$. Then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}_W$ is split semisimple.

Note that Tits' deformation theorem, [41, Theorem 7.2], implies that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\mathsf{Irr}(W) \leftrightarrow \mathsf{Irr}\, K(\mathbf{v})\mathcal{H}_W$.

Remark 7.1. When $W = G(m, 1, n)$ the set $\mathcal{K}/W$ is $\{\mathcal{R}, \mathcal{S}\}$ where $\mathcal{R}$ is the orbit of hyperplanes that define the reflections in the conjugacy class R and $\mathcal{S}$ is the orbit of hyperplanes defining the reflections in the conjugacy classes $S_0, \ldots, S_{m-1}$. Therefore $e_\mathcal{R} = 2$ and $e_\mathcal{S} = m$. Similarly, when $W = G(m, d, n)$ and $n \neq 2$ or $n = 2$ and p odd the set $\mathcal{K}/W$ is $\{\mathcal{R}, \mathcal{S}\}$ where $\mathcal{R}$ is the orbit of hyperplanes that define the reflections in the conjugacy class R and $\mathcal{S}$ is the orbit of hyperplanes defining the reflections in the conjugacy classes $S_d, \ldots, S_{d(p-1)}$. Therefore $e_\mathcal{R} = 2$ and $e_\mathcal{S} = p$. However, when $W = G(m, d, 2)$ with d even, the set $\mathcal{K}/W$ is $\{\mathcal{R}_1, \mathcal{R}_2, \mathcal{S}\}$, where $\mathcal{R}_1$, $\mathcal{R}_2$ are the orbits of the hyperplanes that define the reflections in the conjugacy classes R_1 and R_2. Here $e_{\mathcal{R}_1} = e_{\mathcal{R}_2} = 2$ and $e_\mathcal{S} = p$.

7.2 Cyclotomic Hecke algebras

The cyclotomic Hecke algebras are certain specializations of the generic Hecke algebra. Let y be an indeterminate.

Definition 7.2. A cyclotomic Hecke algebra is the $\mathbb{Z}_K[y, y^{-1}]$-algebra induced from $\mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]\mathcal{H}_W$ by an algebra homomorphism of the form

$$\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}] \to \mathbb{Z}_K[y, y^{-1}], \qquad v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}},$$

where the tuple $\mathbf{n} := \{(n_{\mathcal{C},j} \in \mathbb{Z}) : \mathcal{C} \in \mathcal{K}/W, 0 \leq j \leq e_\mathcal{C} - 1\}$ is chosen such that the following property holds. Set $x := y^{|\mu(K)|}$ and let z be an indeterminate. Then the element of $\mathbb{Z}_K[y, z]$ defined by

$$\Gamma_\mathcal{C}(y, z) = \prod_{j=0}^{e_\mathcal{C}-1} (z - \eta_{e_\mathcal{C}}^j y^{n_{\mathcal{C},j}})$$

is required to be invariant under $\mathrm{Gal}\,(K(y)/K(x))$ for all $\mathcal{C} \in \mathcal{K}/W$. In other words, $\Gamma_\mathcal{C}(y, z)$ is contained in $\mathbb{Z}_K[x^{\pm 1}, z]$. The cyclotomic Hecke algebra corresponding to $\mathbf{n}$ is denoted $\mathcal{H}_W(\mathbf{n})$.

The symmetric form t on $\mathcal{H}_W$ induces a symmetrizing form on $K(y)\mathcal{H}_W(\mathbf{n})$ and this algebra is split semisimple by [24, (4.3)]. Therefore Tits' deformation theorem implies that we have bijections

$$\mathsf{Irr}(W) \leftrightarrow \mathsf{Irr}\, K(y)\mathcal{H}_W(\mathbf{n}) \leftrightarrow K(\mathbf{v})\mathcal{H}_W.$$

7.3 Rouquier families

The *Rouquier ring* is defined to be $\mathcal{R}(y) = \mathbb{Z}_K[y, y^{-1}, (y^n - 1)^{-1} \; : \; n \in \mathbb{N}]$. Since $\mathcal{H}_W$ is free of rank $|W|$, $\mathcal{R}(y)\mathcal{H}_W(\mathbf{n}) \subset K(y)\mathcal{H}_W(\mathbf{n})$ is also free of rank $|W|$. We define an equivalence relation on $\mathsf{Irr}\, K(y)\mathcal{H}_W(\mathbf{n}) = \mathsf{Irr}(W)$ by saying that $\lambda \sim \mu$ if and only if λ and μ belong to the same block of $\mathcal{R}(y)\mathcal{H}_W(\mathbf{n})$. The equivalence classes of this relation are called *Rouquier families.*

Fix a parameter $\mathbf{c}$ for $G(m,d,n)$ that extends to a parameter $\mathbf{c}$ for $G(m,1,n)$, translated into the form $\mathbf{h} = (h, H_0, \ldots, H_{m-1})$ as described in (6.4). Again we make the assumption that $h = -1$ and $\mathbf{h} \in \mathbb{Q}^{m+1}$. Choose $e \in \mathbb{N}$ such that eh and $eH_i \in \mathbb{Z}$ for all $0 \le i \le m-1$. Then $\mathbf{n} = (n_{\mathcal{R},0}, n_{\mathcal{R},1}, n_{\mathcal{S},0}, \ldots, n_{\mathcal{S},m-1})$ is fixed to be $n_{\mathcal{R},0} = e, n_{\mathcal{R},1} = 0$ and $n_{\mathcal{S},j} = e\sum_{i=1}^{j} H_i$ for $0 \le j \le m-1$. From now on we fix $K = \mathbb{Q}(\eta_m)$ and $\mathbb{Z}_K = \mathbb{Z}[\eta_m]$. Recall the morphism Υ defined in (2.4).

Conjecture 1 (Martino, [73], (2.7))**.** *Let* $\mathbf{c}, \mathbf{h}$ *and* $\mathbf{n}$ *be as above.*

1. *The partition of* $\mathsf{Irr}\, G(m,d,n)$ *into Rouquier families associated to* $\mathcal{H}_{G(m,d,n)}(\mathbf{n})$ *refines the* $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ *partition. For generic values of* $\mathbf{c}$ *the partitions are equal.*

2. *Let* $q \in \Upsilon^{-1}(0)$ *and let* $K(y)B_1 \oplus \cdots \oplus K(y)B_k$ *be the sum of the corresponding Rouquier blocks. Then* $\dim(\mathbb{C}[\Upsilon^*(0)_q]) = \dim_{K(y)} K(y)B_1 \oplus \cdots \oplus K(y)B_k$.

The Rouquier families for $G(m,1,n)$ are calculated by Chlouveraki [23] using the idea of *essential hyperplanes.* The essential hyperplanes for $G(m,1,n)$ in $\mathbb{Z}^{m+1}$ are of the form $(kn_{\mathcal{R},0} + n_{\mathcal{S},i} - n_{\mathcal{S},j} = 0)$ for $0 \le i < j \le m-1$ and $-m < k < m$, and $(n_{\mathcal{R},0} = 0)$.

Definition 7.3. Let $\mathbf{n} \in \mathbb{Z}^{m+1}$.

- The hyperplane $(kn_{\mathcal{R},0} + n_{\mathcal{S},i} - n_{\mathcal{S},j} = 0)$ is said to be *essential* if there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\eta_m]$ such that $\eta_m^i - \eta_m^j \in \mathfrak{p}$. The hyperplane $(N = 0)$ is always assumed to be essential.

- If $\mathbf{n}$ belongs to the essential hyperplane $(kn_{\mathcal{R},0} + n_{\mathcal{S},i} - n_{\mathcal{S},j} = 0)$ and $\mathbf{n}$ does not belong to any other essential hyperplane then $\mathbf{n}$ is said to be a *generic* element of $(kn_{\mathcal{R},0} + n_{\mathcal{S},i} - n_{\mathcal{S},j} = 0)$.

If $\mathbf{n} \in \mathbb{Z}^{m+1}$ does not belong to any essential hyperplane then the corresponding Rouquier families are independent of the choice of $\mathbf{n}$. Similarly, if $\mathbf{n}$ is a generic element in some essential hyperplane then the Rouquier families for $\mathbf{n}$ are independent of the choice of $\mathbf{n}$. A general element $\mathbf{n} \in \mathbb{Z}^{m+1}$ will belong to a collection of essential hyperplanes $H_1, \ldots, H_k = 0$. It has been shown by Chlouveraki [24] that Rouquier families have the property of *semi-continuity.* This means that the partition of $\mathsf{Irr}\, G(m,1,n)$ into Rouquier families for $\mathbf{n}$ is the finest partition of $\mathsf{Irr}\, G(m,1,n)$ that is refined by the Rouquier families partition of $\mathsf{Irr}\, G(m,1,n)$ associated to each of the essential hyperplanes $H_i = 0$. Therefore if $\underline{\lambda}$ and $\underline{\mu}$ are in the same Rouquier family for some essential hyperplane $H_i = 0$ then they are in the same Rouquier family for $\mathbf{n}$.

Proposition 7.4 ([23], Proposition 3.15)**.** *Let* $(n_{\mathcal{S},i} - n_{\mathcal{S},j} = 0)$ *be an essential hyperplane and choose* $\mathbf{n}$ *to be a generic element of* $(n_{\mathcal{S},i} - n_{\mathcal{S},j} = 0)$*. Then* $\underline{\lambda}, \underline{\mu} \in \mathcal{P}(n,m)$ *are in the same Rouquier family of* $\mathcal{R}(y)\mathcal{H}_{G(m,1,n)}(\mathbf{n})$ *if and only if*

1. $\lambda^a = \mu^a$ *for all* $a \neq s, t$; *and*

2. $Res_{(\lambda^s,\lambda^t)}(x) = Res_{(\mu^s,\mu^t)}(x)$.

Proof. The result [23, Proposition 3.15] is stated in terms of weighted content but [12, Proposition 3.4] shows that we can reformulate the result in terms of residues. The weighting is $(0, k)$, which in our case becomes $(0, 0)$ since $k = 0$. $\square$

Lemma 7.5. *Let* $\underline{\lambda}, \underline{\mu} \in \mathcal{P}(m, n)$. *We write* $\underline{\lambda} \sim \underline{\mu}$ *if there exists* $0 \leq i \leq p-1$ *and* $0 \leq j < k \leq d-1$ *such that* $\lambda^a = \mu^a$ *for all* $a \neq i+jp, i+kp$ *and*

$$Res_{(\lambda^{i+jp},\lambda^{i+kp})}(x) = Res_{(\mu^{i+jp},\mu^{i+kp})}(x).$$

Now choose $\mathbf{n}$ *to be a generic parameter for* $\mathcal{H}_{G(m,d,n)}$. *Then the partition of* $\mathsf{Irr}\, G(m,1,n)$ *into Rouquier families for* $\mathcal{R}(y)\mathcal{H}_{G(m,1,n)}(\mathbf{n})$ *is the set of equivalence classes in* $\mathsf{Irr}\, G(m,1,n)$ *under the transitive closure of* $\sim$.

Proof. The only hyperplanes that might be essential for $\mathbf{n}$ are of the form $(n_{\mathcal{S},i+jp} - n_{\mathcal{S},i+kp} = 0)$ for $0 \leq i \leq p-1$ and $0 \leq j < k \leq d-1$. However not all of these hyperplanes will be essential. Let us say that the m-multi-partition $\underline{\lambda}$ is *linked* to the m-multi-partition $\underline{\mu}$ if there exists an essential hyperplane $(n_{\mathcal{S},i+jp} - n_{\mathcal{S},i+kp} = 0)$ containing $\mathbf{n}$ such that

$$\mathrm{Res}_{(\lambda^{i+jp},\lambda^{i+kp})}(x) = \mathrm{Res}_{(\mu^{i+jp},\mu^{i+kp})}(x).$$

Then, by Proposition 7.4 and the principal of semi-continuity, the Rouquier families for $\mathcal{R}(y)\mathcal{H}_{G(m,1,n)}(\mathbf{n})$ are the set of equivalence classes in $\mathsf{Irr}\, G(m,1,n)$ under the transitive closure of "linked". Since $\underline{\lambda}$ linked $\underline{\mu}$ implies that $\underline{\lambda} \sim \underline{\mu}$, the Rouquier families refine the partition defined by $\sim$. Therefore we must show that if $\underline{\lambda} \sim \underline{\mu}$ (via $i+jp, i+kp$ say) then there exists a chain of m-multi-partitions $\underline{\lambda} = \underline{\lambda}_1, \ldots, \underline{\lambda}_q = \underline{\mu}$ such that $\underline{\lambda}_\alpha$ is linked to $\underline{\lambda}_{\alpha+1}$ for all $1 \leq \alpha \leq q-1$. For each $0 \leq i \leq p-1$ and $0 \leq j \leq d-1$, the result [22, Lemma 3.6] says that the multi-partitions $\underline{\lambda}$ and $(i, i+jp) \cdot \underline{\lambda}$ belong to the same Rouquier family for $\mathcal{R}(y)\mathcal{H}_{G(m,1,n)}(\mathbf{n})$, where $(i, i+jp)$ is the transposition swapping the partitions λ^i and λ^{i+jp}. In particular, this result (assuming that $d > 1$) shows that there exists some $l \neq 0$ such that the hyperplane $(n_{\mathcal{S},i} - n_{\mathcal{S},i+lp} = 0)$ is essential. Applying the result [22, Lemma 3.6], we see that $\underline{\lambda}$ is in the same Rouquier family as

$$\underline{\lambda}' := (i, i+kp) \cdot (i+lp, i+jp) \cdot \underline{\lambda}$$

and $\underline{\mu}$ is in the same Rouquier family as

$$\underline{\mu}' := (i, i+kp) \cdot (i+lp, i+jp) \cdot \underline{\mu}.$$

Now $(\lambda')^a = (\mu')^a$ for all $a \neq i, i+lp$ and

$$\mathrm{Res}_{((\lambda')^i,\lambda^{i+lp})}(x) = \mathrm{Res}_{((\mu')^i,\mu^{i+lp})}(x).$$

Since the hyperplane $(n_{\mathcal{S},i} - n_{\mathcal{S},i+lp} = 0)$ is essential, this implies that $\underline{\lambda}'$ is linked to $\underline{\mu}'$ and there must be a chain from $\underline{\lambda}$ to $\underline{\mu}$ as required. $\square$

We will require the following combinatorial result. The proof uses the representation theory of cyclotomic Hecke algebras, it would be interesting to have a direct combinatorial proof.

Lemma 7.6. *Let $\underline{\lambda}$ and $\underline{\mu}$ be two m-multi-partitions of n. Then $Res_{\underline{\lambda}}(x) = Res_{\underline{\mu}}(x)$ if and only if there exist $\underline{\lambda} = \underline{\lambda}(1), \dots, \underline{\lambda}(k) = \underline{\mu} \in \mathcal{P}(m,n)$ and $s(i) \neq t(i) \in \{1,\dots,m\}$, $1 < i \leq k$, such that*

1. *$\lambda(i-1)^a = \lambda(i)^a$ for all $a \neq s(i), t(i)$; and*
2. *$Res_{(\lambda(i-1)^{s(i)},\lambda(i-1)^{t(i)})}(x) = Res_{(\lambda(i-1)^{s(i)},\lambda(i-1)^{t(i)})}(x), \quad \forall\, 1 < i \leq k.$*

Proof. Let us fix $\mathbf{n} = (n_{\mathcal{R},0}, n_{\mathcal{R},1}, n_{\mathcal{S},0}, \dots, n_{\mathcal{S},m-1})$ with $n_{\mathcal{R},0} = 1$,$n_{\mathcal{R},1} = 0$ and $n_{\mathcal{S},i} = 0$ for all $0 \leq i \leq m-1$. Then the Lemma is the result [23, Proposition 3.19] for our special parameter $\mathbf{n}$, noting once again that [12, Proposition 3.4] allows us to rephrase [23, Proposition 3.19], which is stated in terms of weighted content, in language of residues. $\square$

We can now confirm the first part of Martino's conjecture for $G(m,d,n)$.

Theorem 7.7. *Let $\mathbf{c} : \mathcal{S}(G(m,d,n)) \to \mathbb{C}$ be a $G(m,1,n)$-equivariant function such that $k = -1$ and $\mathbf{h} \in \mathbb{Q}^{m+1}$. Choose $e \in \mathbb{N}$ such that eh and $eH_i \in \mathbb{Z}$ for all $0 \leq i \leq m-1$. Fix $n_{\mathcal{R},0} = e, n_{\mathcal{R},1} = 0$ and $n_{\mathcal{S},j} = e\sum_{i=1}^{j} H_i$ for $0 \leq j \leq m-1$. Then*

1. *the partition of $\mathsf{Irr}\, G(m,d,n)$ into Rouquier families associated to $\mathcal{H}_{G(m,d,n)}(\mathbf{n})$ refines the $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ partition;*
2. *the partition of $\mathsf{Irr}\, G(m,d,n)$ into Rouquier families associated to $\mathcal{H}_{G(m,d,n)}(\mathbf{n})$ equals the $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ partition for generic values of the parameter $\mathbf{c}$.*

Proof. It is shown in [22, Theorem 3.10] that if $\underline{\lambda}$ is a d-stuttering m-multi-partition of n such that $\{\underline{\lambda}\}$ is a Rouquier family for $\mathcal{R}(y)\mathcal{H}_{G(m,1,n)}(\mathbf{n})$ then the sets $\{(\underline{\lambda}, \epsilon)\}$, $\epsilon \in C_d^{\vee}$, are Rouquier families for $\mathcal{R}(y)\mathcal{H}_{G(m,1,n)}(\mathbf{n})$. This agrees with Theorem 6.29 (1). The second part of [22, Theorem 3.10] shows that if $\mathcal{P}$ is a Rouquier family for $\mathcal{R}(y)\mathcal{H}_{G(m,1,n)}(\mathbf{n})$ not of the type just described then, in the notation of Theorem 6.21, $\Gamma(\mathcal{P})$ is a Rouquier family for $\mathcal{R}(y)\mathcal{H}_{G(m,d,n)}(\mathbf{n})$. The result [73, Corollary 3.13] shows that the partition of $\mathsf{Irr}\, G(m,1,n)$ into Rouquier families associated to $\mathcal{H}_{G(m,1,n)}(\mathbf{n})$ refines the $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$ partition. Therefore there exists a $\mathsf{CM}_{\mathbf{c}}(G(m,1,n))$-partition $\mathcal{Q}$ such that $\mathcal{P} \subseteq \mathcal{Q}$. By Theorem 6.29 (2), $\Gamma(Q)$ is a $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$-partition. Thus $\Gamma(\mathcal{P}) \subseteq \Gamma(\mathcal{Q})$ implies that the partition of $\mathsf{Irr}\, G(m,d,n)$ into Rouquier families refines the $\mathsf{CM}_{\mathbf{c}}(G(m,d,n))$ partition.

Now let $\mathbf{c}$ be a generic parameter for the rational Cherednik algebra associated to $G(m,d,n)$. We think of $\mathbf{c}$ as a parameter for the rational Cherednik algebra associated to $G(m,1,n)$. Thus it is a generic point of the subspace defined by $c_j = 0$ for all $j \not\equiv 0 \bmod d$. Correspondingly, $\mathbf{n}$ is a generic point in the sublattice of $\mathbb{Z}^{m+1}$ defined by the equations $n_{\mathcal{S},i+jp} - n_{\mathcal{S},i+kp} = 0$ for $0 \leq i \leq p-1$ and $0 \leq j < k \leq d-1$. We wish to show that the Calogero-Moser partition of $\mathsf{Irr}\, G(m,d,n)$ equals the partition of $\mathsf{Irr}\, G(m,d,n)$ into Rouquier families. As explained in the previous paragraph, [22, Theorem 3.10] and Theorem 6.29

imply that it suffices to show that the Calogero-Moser partition of $\mathsf{Irr}\, G(m,1,n)$ for $\mathbf{c}$ equals the partition of $\mathsf{Irr}\, G(m,1,n)$ into Rouquier families for $\mathbf{n}$. The proof of Lemma 6.31 shows that $\underline{\lambda}, \underline{\mu} \in \mathcal{P}(m,n)$ are in the same Calogero-Moser partition of $\mathsf{Irr}\, G(m,1,n)$ if and only if

$$\sum_{j=0}^{d-1} \mathrm{Res}_{\lambda^{i+pj}}(x^e) = \sum_{j=0}^{d-1} \mathrm{Res}_{\mu^{i+pj}}(x^e) \quad \forall\, 0 \leq i \leq p-1.$$

Combining the results Lemma 7.4 and Lemma 7.6 shows that $\underline{\lambda}, \underline{\mu} \in \mathcal{P}(m,n)$ are in the same Rouquier family of $\mathcal{R}(y)\mathcal{H}_{G(m,1,n)}(\mathbf{n})$ if and only if the same condition holds. $\square$

7.4 Remarks

1. The main result of this chapter has appeared in the preprint [4].
2. The book [24] is a very comprehensive reference on Rouquier families and contains a description of the families for all complex reflection groups.

Appendix A

Calculations and GAP code

A.1 Calculations for Lemma 5.13

In this section we use freely the notation employed in Lemma 5.13. The aim of the calculation is to show that the map η is both a left and right inverse to ζ. We fix coset representatives $w_1, \ldots, w_k$ of H in G, where $k = \frac{|G|}{|H|}$. Let us begin by recalling the definition of the maps ζ and η. ζ is the map from $C(G, H, A) \cdot \iota(\mathbf{e}_G)$ to $\mathrm{Fun}_H(G, A\mathbf{e}_H)$ defined by

$$\zeta : \quad M \cdot \iota(\mathbf{e}_G) \mapsto M(\delta)$$

where $M \in C(G, H, A)$ and $\delta \in \mathrm{Fun}_H(G, A)$ is the function $\delta(g) = \mathbf{e}_H, \forall\, g \in G$. Then η is the map in the opposite direction defined by

$$\eta : \quad f \mapsto \left(h(-) \mapsto f(-) \sum_{g \in G} h(g) \right),$$

where $f \in \mathrm{Fun}_H(G, A\mathbf{e}_H)$ and $h \in \mathrm{Fun}_H(G, A)$. With respect to our coset representatives, $M = (m_{ij})_{1 \leq i,j, \leq k}$ where $m_{ij} \in A$. If $f = (f_1, \ldots, f_k) \in \mathrm{Fun}_H(G, A)$, then $(Mf)(w_i) = \sum_{j=1}^{k} m_{ij} f_j$ and

$$\iota(\mathbf{e}_G) \cdot f(w_i) = \left(\sum_{h \in H} h \right) \left(\sum_{j=1}^{k} f(w_j) \right) = |H| \cdot \mathbf{e}_H \sum_{j=1}^{k} f_j.$$

Hence

$$(M\iota(\mathbf{e}_G)) \cdot f(w_i) = \sum_{j=1}^{k} m_{ij} (\iota(\mathbf{e}_G f))_j = \sum_{j=1}^{k} m_{ij} \sum_{l=1}^{k} |H| \cdot \mathbf{e}_H f_l = \left(\sum_{j=1}^{k} m_{ij} \right) |H| \cdot \mathbf{e}_H \left(\sum_{l=1}^{k} f_l \right). \quad \text{(A.1)}$$

We see that

$$\zeta(M\iota(\mathbf{e}_G)) = M(\delta)(w_i) = \sum_{j=1}^{k} m_{ij} \delta_j = \sum_{j=1}^{k} m_{ij} \mathbf{e}_H, \quad \text{(A.2)}$$

and, for $h = (h_1, \ldots, h_k) \in \operatorname{Fun}_H(G, A\mathbf{e}_H)$

$$\eta(f)(h(w_i)) = f_i \sum_{g\in G} h(g) = |H|\, f_i\, \mathbf{e}_H \sum_{j=1}^{k} h_j. \tag{A.3}$$

Now we check that $\eta \circ \zeta = id$ and $\zeta \circ \eta = id$. Using (A.2) and (A.3) we have

$$\eta \circ \zeta(M \cdot \iota(\mathbf{e}_G))(f)(w_i) = \zeta(M\iota(\mathbf{e}_G))(w_i) \cdot |H| \cdot \mathbf{e}_H \sum_{j=1}^{k} f_j = |H| \cdot (\zeta(M\iota(\mathbf{e}_G)))_i \cdot \mathbf{e}_H \sum_{j=1}^{k} f_j$$

$$= |H| \cdot \left(\sum_{l=1}^{k} m_{il}\mathbf{e}_H\right) \cdot \mathbf{e}_H \left(\sum_{j=1}^{k} f_j\right) = \left(\sum_{l=1}^{k} m_{il}\right) \cdot |H| \cdot \mathbf{e}_H \left(\sum_{j=1}^{k} f_j\right) = M \cdot \iota(\mathbf{e}_G)(f)(w_i),$$

hence $\eta \circ \zeta = id$. For $f \in \operatorname{Fun}_H(G, A\mathbf{e}_H)$, define $M_f \in C(G, H, A)$ by

$$M_f = \left(\frac{|H|}{|G|} f_i\right)_{i,j}.$$

Then, using (A.1),

$$\eta(f) = M_f\, \iota(\,\mathbf{e}_G).$$

Now

$$\zeta \circ \eta(f)(w_i) = \zeta(M_f\iota(\mathbf{e}_G))(w_i) = \sum_{j=1}^{k} m_{ij}\mathbf{e}_H = \left(\sum_{l=1}^{k} \frac{|H|}{|G|} f_i\right) \mathbf{e}_H = f_i \cdot \mathbf{e}_H = f_i = f(w_i),$$

hence

$$\zeta \circ \eta(f) = f.$$

A.2 Calculations for Corollary 5.21

We now describe the embedding alluded to in Corollary 5.21. It will be done in slightly greater generality than in the setup of Corollary 5.21 so that the presentation is clearer. We use freely the notation of (5.5). Let B be an algebra with a fixed isomorphism $\theta : B \longrightarrow C(G, H, A)$ and embedding $\iota_1 : \mathbb{C}G \hookrightarrow B$ such that the composition $\theta \circ \iota_1 : \mathbb{C}G \hookrightarrow C(G, H, A)$ equals the embedding $\iota : \mathbb{C}G \hookrightarrow C(G, H, A)$ defined in (5.5). Then we wish to show that there exists an embedding $j : \mathbf{e}_H A \mathbf{e}_H \hookrightarrow C(G, H, A)$ such that $\theta(\mathbf{e}_G B \mathbf{e}_G) = j(\mathbf{e}_H A \mathbf{e}_H)$ and splitting $j^{-1} : C(G, H, A) \to \mathbf{e}_H A \mathbf{e}_H$ of j such that

$$j^{-1} \circ \theta : \mathbf{e}_G B \mathbf{e}_G \longrightarrow \mathbf{e}_H A \mathbf{e}_H$$

is a well defined isomorphism. Once again we fix coset representatives $w_1, \ldots, w_k$ of H in G, where $k = \frac{|G|}{|H|}$. Let $\delta \in \operatorname{Fun}_H(G, A\mathbf{e}_H)$ be as in (A.1) above and denote by 1_G the identity element in G. The map j is given by $\mathbf{e}_H a \mathbf{e}_H \mapsto j(\mathbf{e}_H a \mathbf{e}_H)$ such that

$$j(\mathbf{e}_H a \mathbf{e}_H)(f)(w) = \mathbf{e}_H a \mathbf{e}_H \cdot \frac{1}{|G|} \sum_{g \in G} f(g), \quad \forall f \in \mathrm{Fun}_H(G, A).$$

It is well defined:

$$j(\mathbf{e}_H a \mathbf{e}_H)(f)(uw) = \mathbf{e}_H a \mathbf{e}_H \cdot \frac{1}{|G|} \sum_{g \in G} f(g) = u \cdot \mathbf{e}_H a \mathbf{e}_H \cdot \frac{1}{|G|} \sum_{g \in G} f(g) = u \cdot j(\mathbf{e}_H a \mathbf{e}_H)(f)(w),$$

for $u \in H, w \in W$ and $f \in \mathrm{Fun}_H(G, A)$. We define the splitting j^{-1} of the map j by

$$j^{-1} \,:\, M \mapsto M(\delta)(1_G).$$

Note that j^{-1} is not an algebra homomorphism. Let us show that j^{-1} is actually a splitting of j as we claim. That is, we wish to show that $j^{-1} \circ j = id_{\mathbf{e}_H A \mathbf{e}_H}$:

$$j^{-1} \circ j(\mathbf{e}_H a \mathbf{e}_H) = j(\mathbf{e}_H a \mathbf{e}_H)(\delta)(1_G) = \mathbf{e}_H a \mathbf{e}_H \cdot \frac{1}{|G|} \sum_{g \in G} \delta(g) = \mathbf{e}_H a \mathbf{e}_H \cdot \frac{1}{|G|} |G| = \mathbf{e}_H a \mathbf{e}_H.$$

Next we wish to derive an explicit expression for $\theta(\mathbf{e}_G b \mathbf{e}_G)$. Let $v_i \in \mathrm{Fun}_H(G, A)$ be defined by $v_i(w_j) = \delta_{ij}$, then

$$\theta(\mathbf{e}_G)(v_i)(w_j) = \frac{1}{|G|} \sum_{g \in G} v_i(g) = \frac{|H|}{|G|} \mathbf{e}_H$$

hence $\theta(\mathbf{e}_G) = \left(\frac{|H|}{|G|} \mathbf{e}_H \right)_{ij}$. Let $b \in B$ and write $\theta(b) = (b_{ij})_{ij}$, then

$$\theta(\mathbf{e}_G b \mathbf{e}_G) = \left(\frac{|H|}{|G|} \mathbf{e}_H \right)_{i,j} \cdot (b_{ij})_{i,j} \cdot \left(\frac{|H|}{|G|} \mathbf{e}_H \right)_{i,j} = \left(\frac{|H|^2}{|G|^2} \mathbf{e}_H \left(\sum_{i,j=1}^{k} b_{ij} \right) \mathbf{e}_H \right)_{i,j}.$$

We now show that $j \circ j^{-1}(\theta(\mathbf{e}_G b \mathbf{e}_G)) = \theta(\mathbf{e}_G b \mathbf{e}_G)$:

$$j^{-1}(\theta(\mathbf{e}_G b \mathbf{e}_G)) \frac{|H|}{|G|} \mathbf{e}_H \left(\sum_{i,j=1}^{k} b_{ij} \right) \mathbf{e}_H$$

and so

$$j \circ j^{-1}(\theta(\mathbf{e}_G b \mathbf{e}_G))(\delta_i)(w_j) = \frac{|H|}{|G|} \mathbf{e}_H \left(\sum_{i,j=1}^{k} b_{ij} \right) \mathbf{e}_H \frac{1}{|G|} \sum_{g \in G} \delta_i(g)$$

$$= \frac{|H|^2}{|G|^2} \mathbf{e}_H \left(\sum_{i,j=1}^{k} b_{ij} \right) \mathbf{e}_H = \theta(\mathbf{e}_G b \mathbf{e}_G)(\delta_i)(w_j)$$

as required.

A.3 Calculations for Lemmata 6.43 and 6.44

In this section we prove Lemmata 6.43 and 6.44. We make the assumption that $m = 2n \geq 6$. If $m = 4$ then $I_2(4)$ is isomorphic to $G(2,1,2)$ (= the Weyl group B_2) and the Calogero-Moser partition can be calculated using Theorems 6.29 and 6.37. We fix a basis x_1, x_2 basis of $\mathfrak{h}^*$ and y_1, y_2 the dual basis of $\mathfrak{h}$ such that

$$a^s \cdot x_1 = \zeta^{-s} x_1,\ a^s \cdot x_2 = \zeta^s x_2,\ a^s \cdot y_1 = \zeta^s y_1,\ a^s \cdot y_2 = \zeta^{-s} \cdot y_2,$$

$$a^s b \cdot x_1 = \zeta^{-s} x_2,\ a^s b \cdot x_2 = \zeta^s x_1,\ a^s b \cdot y_1 = \zeta^s y_2,\ a^s b \cdot y_2 = \zeta^{-s} y_1.$$

We take $\alpha_{a^s b} = x_1 - \zeta^{-s} x_2$ and $\alpha_{a^s b}^\vee = y_1 - \zeta^s y_2$ so that $(\alpha_{a^s b}, \alpha_{a^s b}^\vee) = 2$. The defining relations for the rational Cherednik algebra $H_{0,(c_1,c_2)}(I_2(m))$ are

$$[x_1, y_1] = \frac{1}{2} c_1 \sum_{i=0}^{n-1} a^{2i} b + \frac{1}{2} c_2 \sum_{i=0}^{n-1} a^{2i+1} b \tag{A.4}$$

$$[x_1, y_2] = \frac{-1}{2} c_1 \sum_{i=0}^{n-1} \zeta^{-2i} a^{2i} b + \frac{-1}{2} c_2 \sum_{i=0}^{n-1} \zeta^{-2i-1} a^{2i+1} b \tag{A.5}$$

$$[x_2, y_1] = \frac{-1}{2} c_1 \sum_{i=0}^{n-1} \zeta^{2i} a^{2i} b + \frac{-1}{2} c_2 \sum_{i=0}^{n-1} \zeta^{2i+1} a^{2i+1} b \tag{A.6}$$

$$[x_2, y_2] = \frac{1}{2} c_1 \sum_{i=0}^{n-1} a^{2i} b + \frac{1}{2} c_2 \sum_{i=0}^{n-1} a^{2i+1} b \tag{A.7}$$

From the character table of $I_2(m)$, given in (6.11), we see that $\mathfrak{h}_i \simeq \mathbb{C} \cdot \{x_1^i, x_2^i\} \simeq \mathbb{C} \cdot \{x_1^{m-i}, x_2^{m-i}\}$. Define

$$\mathsf{eu} := x_1 y_1 + y_1 x_1 + x_2 y_2 + y_2 x_2 = 2(x_1 y_1 + x_2 y_2) + \mathsf{z}$$

where

$$\mathsf{z} = -c_1 \sum_{i=0}^{n-1} a^{2i} b - c_2 \sum_{i=0}^{n-1} a^{2i+1} b \in Z(I_2(m)).$$

A direct calculation using the relations (A.4 - A.7) shows that $\mathsf{eu} \in Z_{\mathbf{c}}(I_2(m))$ for all $\mathbf{c}$. It will act on $L(\lambda)$ as some scalar $\mathsf{eu}(\lambda)$. It will act on the baby Verma module $\Delta(\lambda)$ as the same scalar $\mathsf{eu}(\lambda)$ and this scalar equals the scalar that the element z acts on λ as. Using the character table of $I_2(m)$ one can calculate that

$$\mathsf{eu}(T) = -n(c_1 + c_2), \quad \mathsf{eu}(S) = n(c_1 + c_2), \quad \mathsf{eu}(V_1) = -n(c_1 - c_2), \quad \mathsf{eu}(V_2) = n(c_1 - c_2)$$

and $\mathsf{eu}(\mathfrak{h}_j) = 0$ for all $1 \leq j \leq n-1$. These values are all different provided $\mathbf{c}$ does not lie on the lines spanned by the vectors $(1,0), (0,1), (1,1)$ and $(1,-1)$. This proves Lemma 6.43.

Now we show that the modules $\mathfrak{h}_1, \ldots, \mathfrak{h}_{n-1}$ always belong to the same block of the Calogero-Moser partition for the dihedral group $I_2(m)$, where $m = 2n$ is even, regardless of the value of the parameter $\mathbf{c}$. This will prove Lemma 6.44. The character table shows that $\mathfrak{h}_{i+1} \simeq \mathbb{C} \cdot \{x_1 \otimes x_1^i, x_2 \otimes x_2^i\} \subset \mathfrak{h}_1 \otimes \mathfrak{h}_i$, for $1 \leq i \leq n-2$. A direct calculation using relations (A.4) and (A.5) shows that $y_1 \cdot x_1 \otimes x_1^i = y_2 \cdot x_1 \otimes x_1^i = 0$ in $\mathfrak{h}_1 \otimes \mathfrak{h}_i \subset \Delta(\mathfrak{h}_i)$, provided $i \leq n-2$. Similarly, relations (A.6) and (A.7) show that $y_1 \cdot x_2 \otimes x_2^i = y_2 \cdot x_2 \otimes x_2^i = 0$ in $\mathfrak{h}_1 \otimes \mathfrak{h}_i \subset \Delta(\mathfrak{h}_i)$, provided $i \leq n-2$. This shows that the isomorphism $\mathfrak{h}_{i+1} \xrightarrow{\sim} \mathfrak{h}_1 \otimes \mathfrak{h}_i$ induces a non-zero $\bar{H}_{\mathbf{c}}(I_2(m))$-morphism $\phi_i : \Delta(\mathfrak{h}_{i+1}) \to \Delta(\mathfrak{h}_i)$ for all $1 \leq i \leq n-2$. Since the baby Verma modules $\Delta(\mathfrak{h}_i)$ are indecomposable with simple head $L(\mathfrak{h}_i)$ this implies that the modules $\mathfrak{h}_1, \ldots, \mathfrak{h}_{n-1}$ all lie in the same block of Calogero-Moser partition.

A.4 GAP code

The code below produces a list called `result` of 34 numbers. These are the 34 numbers appearing in the bottom row of table 3.5.

```
RequirePackage( "chevie" );
q := X( Cyclotomics );; q.name := "q";;

PolynomialCoefficients:= function(f)
     local coes,g,d,c,i;
     coes := [];
     g := f;
     while Degree(g) > 0 do
          d := Degree(g);
          c := LeadingCoefficient(g);
          coes[d+1] := c;
          g := g - c*q^d;
     od;
     coes[1] := LeadingCoefficient(g);
     for i in [1..Length(coes)] do
          if IsBound(coes[i]) = false then
                 coes[i] := 0;
          fi;
     od;
     return coes;
end;

TrailingTerm:= function(f)
     local coes,p,i;
     coes := PolynomialCoefficients(f);
     p := PositionProperty(coes, i -> not( i = 0));
     return coes[p]*q^(p-1);
end;
```

```
result := [];
for n in [4..37] do
     G := ComplexReflectionGroup(n);
     O := Size(G);
     fakedegrees := FakeDegrees(G,q);
     N := Length(fakedegrees);
     exponents := ReflectionDegrees(G);
     PoincarePolynomial := Product(List([1..Length(exponents)],
                              i -> ((1-q^exponents[i]) / (1-q))));
     test := List([1..N], i -> EuclideanRemainder(TrailingTerm(
             fakedegrees[i])*PoincarePolynomial,fakedegrees[i]));
     Add(result,Length(Filtered(test, i -> not( i = 0*q^0))));
od;
```

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